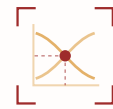


FSP Training 08

Preparation for Mock Exam 09



Instructions

This training page covers all of Section 08 and prepares you for Mock Exam 09. Mock Exam 09 is a full recap in the format of the real FSP: Problem 1 on cost and profit (37 points), Problem 2 on curve sketching with integration (38 points) and Problem 3 on a screening with the binomial distribution (25 points). Set A trains setting up: you translate a text into mathematics and check your translation. It mixes Section 08 with the part types of Mock Exam 09. Set B is a practice mock in the format of Mock Exam 09: 180 minutes plus 5 minutes reading time, 100 points, and all 3 problems must be completed. Write down every step as you would in the exam. The points show what each part is worth.

Keep intermediate results unrounded and type the exact rate per period into the calculator, e.g. $0.036/12$. Give amounts in EUR to the cent and interest rates in percent with two decimal places. A number of payments or years that must reach a target is rounded up: check the whole numbers on both sides. A payment or an amount that must be reached is a minimum, so round it up to the cent. A largest affordable amount is a maximum, so round it down. A lower limit price is a minimum price: round it up to the cent if you have to round. Round probabilities to four decimal places and other results to two decimal places. A minimum number n is always rounded up: check $n - 1$ and n .

In the real FSP, decimals are written with a comma (1,25 means 1.25).

Set A: Set Up & Check (50 minutes)

From Text to Mathematics (xx)

Translate each statement into mathematics. Then answer the check question: it tells you whether your translation works.

1. A tattoo studio wants to have 9,500 EUR in 3 years for new equipment. Its bank pays 2.25 % per year, compounded monthly. **Compute** how much the studio must deposit today and **write** the formula you used.

Check: A classmate says: "Compounding monthly or yearly makes no difference for the deposit." **Compute** the amount for yearly compounding and **explain** the difference.

2. A company puts 45,000 EUR into a fund that pays 3.5 % per year, compounded continuously. **Write** the equation for the time after which the capital has grown to 54,000 EUR and **solve** it. **Compute** the effective annual rate of the fund.

Check: A classmate solves $45000 \cdot 1.035^t = 54000$ and gets about 5.30 years. **Explain** which compounding this equation assumes and why its answer is longer.

3. A market trader rents a stall in a covered market for 4 years and pays 640 EUR rent at the beginning of each month. The trader calculates with 3 % per year, compounded monthly. **Decide** which annuity formula fits, **write** it and **compute** what the rent payments are worth today.

Check: A classmate answers 30,720 EUR. **Explain** what this number is and why the value today must be smaller.

4. A company borrows 48,000 EUR for 5 years at 6.6 % per year, compounded monthly, and repays the loan with equal payments at the end of each month. **Compute** the monthly payment. **Write** the first two rows of the amortization schedule (payment, interest, principal, balance).

Check: A classmate computes the interest of the first month as $48000 \cdot 0.066 = 3,168$ EUR. **Explain** the mistake and **check** your first row: interest and principal must add up to the payment.

5. A taxi company wants to buy an electric taxi for 64,000 EUR in 5 years. It deposits the same amount at the end of each month into a savings account that pays 2.9 % per year, compounded monthly. **Write** the formula for the monthly deposit and **compute** it.

Check: A colleague divides $64000 : 60 \approx 1,066.67$ EUR. **Explain** why the correct deposit is smaller, and **compute** how much of the 64,000 EUR the bank pays as interest.

6. A workshop makes wooden chopping boards. Its daily costs are $C(x) = 864 + 15x + 0.06x^2$ in EUR for x boards. **Compute** the short-term and the long-term lower limit price. A shop chain pays 21 EUR per board. **Decide** what the workshop should do in the short run and in the long run.

Check: A colleague says: “21 EUR lies below our lowest unit costs (average total cost), so we must stop production at once.” **Explain** what is wrong with this argument.

7. The total costs of a company are a polynomial of third degree, $C(x) = ax^3 + bx^2 + cx + d$, with C in MU and x in QU. **Translate** each sentence into an equation for a , b , c and d . Do not solve the system.

(1) The fixed costs are 200 MU.

(2) At 4.5 QU, the curvature of the cost function changes sign (the graph of C has an inflection point there).

(3) At 3 QU, the marginal costs are 42 MU per QU.

(4) An output of 6 QU costs 560 MU in total.

Check: **Show** that $C(x) = 2x^3 - 27x^2 + 150x + 200$ meets all four equations. A classmate writes sentence (2) as $C'(4.5) = 0$. **Explain** what this equation would say instead.

8. An FSP part says: “Determine the nature and the coordinates of all stationary points.” **Write** the necessary and the sufficient condition you use for $f(x) = 2x^3 - 9x^2 - 60x + 20$, and **carry** them out.

Check: A classmate answers: “Maximum at $x = -2$, minimum at $x = 5$.” **Explain** why this answer does not earn full points.

9. A vet tests cats for a kidney disease with a blood test. 8 % of the tested cats have the disease (K): this is the prevalence (base rate). The test is positive (+) for 90 % of the cats with the disease: this is its sensitivity (true positive rate). It is negative (–) for 94 % of the healthy cats: this is its specificity (true negative rate). **Write** the three rates as probabilities. **Write** “a cat with a positive test result does indeed have the disease” and “a cat has the disease but is incorrectly identified as healthy” as probabilities and **compute** both, e.g. with a table for 2,500 cats.

Check: A classmate writes the specificity as $P(+ | \bar{K}) = 0.94$. **Explain** what this probability would mean and **give** its correct value.

10. In a large call centre, 1.5 % of all calls are cut off by a technical fault, independently of each other. X is the number of cut-off calls among 600 calls. **Write** each event with values $P(X \leq k)$ of the cumulative binomial distribution and **compute** it with the calculator: (1) not more than 7 calls are cut off, (2) at least 10 calls are cut off, (3) more than 6 but fewer than 12 calls are cut off.

Check: A classmate writes event (3) as $P(X \leq 12) - P(X \leq 6)$. **List** the values of X in the classmate’s event and in event (3).

One Paragraph, All Conditions (xx)

A company builds electric kilns (ovens for firing clay) and sells them to pottery studios at a fixed market price of 30 MU per QU (1 QU = 10 kilns, 1 MU = 1,000 EUR). Its monthly total costs are a polynomial of third degree in the output x in QU. The fixed costs are 45 MU. At 9 QU the marginal costs are lowest, at 12 MU per QU. An output of 10 QU costs 311 MU in total. For a new workshop, the company takes out a loan of 240,000 EUR at 5.1 % per year, compounded monthly, and repays it with equal payments at the end of each month over 6 years. The management wants to know the cost function, the output with the lowest unit costs and the long term price floor (the long-term lower limit price), the break-even point and the profit limit (the upper break-even point) at the market price, and the largest monthly profit. It also asks for the monthly loan payment, whether the largest monthly profit covers it, and how much of the loan is still owed after 3 years.

List every given value and every question in this paragraph as a mathematical statement: an equation, a function to minimise or maximise, or a formula. **Name** the rule or the formula you would use. Do not solve anything.

Check: An assistant writes “At 9 QU the marginal costs are lowest” as $C'(9) = 0$. **Explain** what went wrong.

FSP Words So Far

The FSP problems use these words from Sections 01 to 08. Learn what each one means and how to write it in mathematics.

Word	Meaning	Mathematics
quantity unit (QU)	a fixed number of items, e.g. 1 QU = 100 items	the output x is counted in QU
monetary unit (MU)	a fixed amount of money, e.g. 1 MU = 100 EUR	results in MU, convert to EUR at the end
fixed costs	costs that arise even if nothing is produced	the constant term of $C(x)$
variable costs	costs that grow with the output	$C(x) - C(0)$ (for linear costs: output times costs per QU)
total costs	fixed costs + variable costs	$C(x)$
revenue	quantity times price	$R(x) = x \cdot \text{price}$
profit	revenue – total costs	$P(x) = R(x) - C(x)$
break-even point	the output at which the profit changes from negative (a loss) to positive	$P(x) = 0$, the smaller zero
profit limit (upper break-even point)	the larger output with zero profit: above it, the profit turns negative again	$P(x) = 0$, the larger zero
profit zone	all outputs with a positive profit	$P(x) > 0$: from the break-even point to the profit limit (or the capacity)
capacity	the maximum possible output	an upper bound: $x \leq \text{capacity}$
growth factor	the factor by which a quantity is multiplied each period	q in (starting value) $\cdot q^t$
growth rate	the change per period in percent	$q = 1 + r$ with r as a decimal, a loss of 4 % gives $q = 0.96$
doubling time	the time after which a quantity has doubled	$q^T = 2$, continuous: $e^{kT} = 2$, so $T = \frac{\ln 2}{k}$
half-life (halving time)	the time after which a quantity has halved	$q^T = 0.5$, continuous: $e^{-kT} = 0.5$, so $T = \frac{\ln 2}{k}$
“deviates from a by at most d ” (tolerance)	the distance from a is at most d	$ x - a \leq d \Leftrightarrow a - d \leq x \leq a + d$

Word	Meaning	Mathematics
“share of A among B ”	the part of B that also belongs to A : the base is B	$\frac{\text{number in } A \cap B}{\text{number in } B}$
“neither A nor B ”	not in A and not in B , i.e. outside both circles	$\overline{A \cup B}$
contingency table	a table of counts for two properties: the inner fields are the regions of the Venn diagram, the margins are the totals	inner fields $A \cap B$, $A \cap \overline{B}$, $\overline{A} \cap B$, $\overline{A} \cap \overline{B}$. Joint probability: field divided by the grand total, conditional probability: field divided by the total of the condition's row or column
gross price, net price	gross price = net price + VAT	with 19 % VAT: gross = $1.19 \cdot$ net
system of linear equations	several linear equations that must hold at the same time	solve by substitution or elimination, check the solution in every equation
no solution, infinitely many solutions	the conditions contradict each other, or one condition only repeats another	elimination leaves $0 = d$ with $d \neq 0$, or $0 = 0$
demand, supply	the quantity buyers want to buy and the quantity sellers offer at a given price	as terms in the price p , e.g. $Q_d(p)$, $Q_s(p)$, or as prices $p_D(x)$, $p_S(x)$ for the quantity x
market equilibrium	the quantity demanded equals the quantity supplied	demand = supply: $Q_d(p) = Q_s(p)$, or $p_D(x) = p_S(x)$. With demand and supply as prices: $D(q) = S(q)$ gives q^* , and $p^* = D(q^*)$
price-demand function	gives the price at which x QU can be sold	$p(x)$, revenue $R(x) = x \cdot p(x)$
monopolist	the only supplier of a product, so it can set the price along the demand	its price follows the price-demand function $p(x)$
average cost (average total cost)	total costs divided by the number of units	$\overline{C}(x) = \frac{C(x)}{x}$ with $x \neq 0$
zero form	all terms on one side, 0 on the other side	$ax^2 + bx + c = 0$, then factor or use the quadratic formula
discriminant	decides how many real solutions a quadratic equation has	$\Delta = b^2 - 4ac$: > 0 two, $= 0$ one, < 0 none

Word	Meaning	Mathematics
biquadratic equation	an equation with only x^4 , x^2 and a constant	substitute $u = x^2$ in $ax^4 + bx^2 + c$, solve for u , then $x = \pm\sqrt{u}$ for $u \geq 0$
domain restrictions (of an equation)	the conditions the unknown must meet so that every term is defined	denominators $\neq 0$, under a root ≥ 0 , inside a logarithm > 0
extraneous solution	a candidate that solves the squared equation but not the original one	check every candidate in the original equation
function	assigns to every input exactly one output	$f(x)$, its graph passes the vertical line test
domain (of a function)	all inputs for which the function is defined, in a context only the sensible ones	e.g. $[0; \text{capacity}]$
range	all outputs that the function takes on its domain	if the function grows or falls: between the values at the two ends, for a parabola also check the vertex
maximum price	the price at which nobody buys any more	$Q_d(p) = 0$, or c in $p = c - dQ$
market saturation	the quantity demanded at a price of 0	a in $Q_d = a - bp$, or the zero of $p(x)$
shortage, surplus	at a price below the equilibrium price, demand exceeds supply (shortage), above it, supply exceeds demand (surplus)	$Q_d(p) > Q_s(p)$, or $Q_d(p) < Q_s(p)$
contribution margin	price minus variable costs per unit: the part of the price that covers the fixed costs	$p - \overline{C}_v(x)$ per unit. Linear costs: break-even quantity = $\frac{\text{fixed costs}}{\text{contribution margin}}$
vertex	the highest or lowest point of a parabola	$x_v = -\frac{b}{2a}$, or (h, k) in $a(x - h)^2 + k$
profit-maximizing output	the output with the highest profit	x_v of $P(x)$, a maximum since $a < 0$, and the price $p(x_v)$. If the capacity lies below x_v : the capacity and its price
shift, stretch, compression	moving or scaling a graph	$a \cdot f(b(x - h)) + k$: first b , then h , then a , then k

Word	Meaning	Mathematics
mirror image at the y -axis	the graph flipped from right to left	$f(-x)$
mirror image at the x -axis	the graph flipped upside down	$-f(x)$
composite function	first the inner function, then the outer function	$(f \circ g)(x) = f(g(x))$
inverse function	reverses a function: gives the input that belongs to an output	solve $y = f(x)$ for x , e.g. $x(p)$ from $p(x)$
polynomial of third degree	a function with powers of x up to x^3	$ax^3 + bx^2 + cx + d$ with $a \neq 0$: four unknowns, so four conditions
degree, leading coefficient	the highest power of x , and the number in front of it	together they decide the end behavior
zero of multiplicity m	the factor $(x - c)$ appears m times in the factored form	odd m : the graph crosses the x -axis at c , even m : it touches it
polynomial long division	divides a polynomial by $(x - c)$ once c is a known zero	$f(x) = (x - c) \cdot q(x)$, the remainder must be 0
asymptotic behavior (end behavior)	what happens to $f(x)$ for $x \rightarrow \infty$ and $x \rightarrow -\infty$	polynomial: degree and sign of the leading coefficient. Rational function: its asymptotes
intercepts	the points where the graph meets the axes	y -intercept $(0, f(0))$, x -intercepts from $f(x) = 0$
rational function	a quotient of two polynomials	$\frac{p(x)}{q(x)}$ with the domain $q(x) \neq 0$
vertical asymptote	a vertical line near which the values grow without bound	a zero of the denominator that does not cancel
hole	a single missing point of the graph	a factor that cancels, its y -value comes from the simplified term
horizontal asymptote	a horizontal line that the graph approaches for $x \rightarrow \pm\infty$	$y = c$ with $f(x) \rightarrow c$
oblique asymptote	a slanted line that the graph approaches for $x \rightarrow \pm\infty$	the linear part of the long division, if the numerator's degree is one higher
grows less than proportionally	grows, but twice the input gives less than twice the output	$a \cdot x^b$ with $0 < b < 1$

Word	Meaning	Mathematics
continuous growth rate	the rate k of a model with the base e	$N_0 \cdot e^{kt}$, growth factor per period e^k
saturation level	a level that a quantity approaches but never exceeds	a horizontal asymptote $y = L$, e.g. of $L - a \cdot e^{-kt}$ or $\frac{L}{1 + Ae^{-kt}}$
behavior for $t \rightarrow \infty$	the value a quantity approaches in the long run	$\lim_{t \rightarrow \infty} N(t)$, often with $e^{-kt} \rightarrow 0$
radian	an angle measure in which a full turn is 2π	$180^\circ = \pi$, arc length $s = r\theta$
amplitude	half the distance between the highest and the lowest value	$ A = \frac{\max - \min}{2}$
midline	the horizontal line halfway between the highest and the lowest value	$y = D = \frac{\max + \min}{2}$
period	the length of one full cycle	$B = \frac{2\pi}{\text{period}}$
phase shift	the horizontal shift of a sine or cosine curve	C in $A \sin(B(t - C)) + D$ or $A \cos(B(t - C)) + D$
mirror image at the midline	the graph flipped upside down about its midline $y = D$	$2D - f(t)$
limit	the value that $f(x)$ approaches when x approaches a number a , or $\pm\infty$	$\lim_{x \rightarrow a} f(x)$, one-sided: $x \rightarrow a^-$ (from the left), $x \rightarrow a^+$ (from the right)
continuous (at a)	the graph has no jump and no gap at a	$f(a)$ is defined, $\lim_{x \rightarrow a} f(x)$ exists, and both are equal
removable discontinuity	a gap that a single value could close, e.g. a hole	the limit exists, but $f(a)$ is not defined or differs from it
derivative	the instantaneous rate of change: the slope of the tangent	$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$
secant, slope of the secant	the line through two points of the graph, and its slope (the average rate of change)	$\frac{f(b) - f(a)}{b - a}$ on $[a; b]$
tangent, tangential gradient	the line that touches the graph at a point, and its slope	$y = f'(a)(x - a) + f(a)$ with the slope $f'(a)$
angle of intersection (with the x -axis)	the angle at which a line with slope m crosses the x -axis	$\alpha = \arctan(m)$, with the indicator D for degrees

Word	Meaning	Mathematics
marginal cost, marginal revenue, marginal profit	about the change of the costs, the revenue or the profit caused by one more unit	$C'(x)$, $R'(x)$, $P'(x) = R'(x) - C'(x)$
stationary point (critical point)	a point of the graph with a horizontal tangent	$f'(x_0) = 0$
nature of a stationary point	whether it is a local maximum, a local minimum or a saddle point	sign change of f' at x_0 , or the sign of $f''(x_0)$
local maximum, local minimum	a point that is higher (lower) than all points nearby	$f'(x_0) = 0$ and $f''(x_0) < 0$ (maximum) or $f''(x_0) > 0$ (minimum)
saddle point	a stationary point at which f' does not change its sign: the graph levels off and keeps its direction	$f'(x_0) = 0$ without a sign change, e.g. $f''(x_0) = 0$ and $f'''(x_0) \neq 0$
necessary condition, sufficient condition	a condition every candidate must meet, and a condition that proves the kind of point ("prove it really is a maximum")	necessary: $f'(x_0) = 0$, sufficient: e.g. $f''(x_0) < 0$ for a maximum
global maximum, global minimum (on an interval)	the highest or lowest value on the whole interval	compare the local extrema with the values at the ends of the interval
concave up, concave down	the graph curves upward like a cup, or downward like a cap	$f''(x) > 0$, or $f''(x) < 0$
inflection point	the point at which the curvature changes sign: the graph is steepest or flattest there	$f''(x_0) = 0$ and $f'''(x_0) \neq 0$, or a sign change of f''
"the curvature changes sign"	the graph changes from concave down to concave up, or back	an inflection point: $f''(x_0) = 0$. For a cubic cost function, the marginal costs are lowest there
inflectional tangent	the tangent at an inflection point: it shows the fastest growth or decline	$y = f'(a)(x - a) + f(a)$ with $f''(a) = 0$
unit costs, lowest unit costs	the total costs per unit (the average total cost), and the output at which they are lowest	$\overline{C}(x) = \frac{C(x)}{x}$ (earlier sheets: $k(x)$), minimum from $\overline{C}'(x) = 0$ and $\overline{C}''(x) > 0$
long term price floor (long-term lower limit price)	the lowest unit costs: below this price, the company cannot cover its total costs in the long run	the minimum value of $\overline{C}(x) = \frac{C(x)}{x}$. Cubic costs: $\overline{C}'(x) = 0$

Word	Meaning	Mathematics
		times x^2 gives a cubic equation
variable costs per unit (variable cost per unit)	the variable costs divided by the output	$\overline{C}_v(x) = \frac{C(x) - C_f}{x}$ (earlier sheets: $k_v(x)$)
short-term lower limit price	the lowest variable costs per unit: below this price, the company cannot even cover its variable costs	the minimum value of $\overline{C}_v(x)$. Quadratic costs: the coefficient of x
function family (German: <i>Funktionsschar</i>)	functions with the same term that differ only in a parameter	$f_a(x)$, e.g. $f_a(x) = x^2 - ax + 1$
parameter	a letter that is fixed within one function of the family but changes from one function to the next	solve a condition for it, e.g. $f'_a(2) = 0$
antiderivative (German: <i>Stammfunktion</i>)	a function whose derivative is f	$F'(x) = f(x)$
constant of integration	the free constant: two antiderivatives of f differ only by a constant	$F(x) + C$, fixed by one condition, e.g. $C(0) = \text{fixed costs}$ or $P(0) = -\text{fixed costs}$
indefinite integral	the family of all antiderivatives	$\int f(x) dx = F(x) + C$
definite integral	a number: the signed area between the graph and the x -axis from a to b	$\int_a^b f(x) dx = F(b) - F(a)$ (Fundamental Theorem of Calculus)
lower limit, upper limit	the start a and the end b of the interval in $\int_a^b f(x) dx$	“in the second month”: from $t = 1$ to $t = 2$
signed area	area above the x -axis counts positive, area below it negative	$\int_a^b f(x) dx$, it can be negative or 0
area (between a graph and the x -axis)	always a positive number	split at the zeros and add the absolute values of the integrals of the pieces
net change	the total change of a quantity, computed from its rate of change	$F(b) - F(a) = \int_a^b F'(x) dx$, e.g. extra costs $\int_a^b C'(x) dx$

Word	Meaning	Mathematics
“total ... within the first ... months”	everything that has come together from the start up to month b	$\int_0^b (\text{rate}) dt$
area enclosed by two graphs	the region between the graphs from one intersection point to the next	$\int_a^b [f(x) - g(x)] dx$ with f on top, limits from $f(x) = g(x)$, split where the graphs cross
“both regions have equal size”	two regions have the same area (not: the integral over both is 0)	compute both areas separately and compare them
consumer surplus	what the buyers gain: they would have paid more than the market price	$CS = \int_0^{q^*} [D(q) - p^*] dq$
producer surplus	what the sellers gain: they would have sold for less than the market price	$PS = \int_0^{q^*} [p^* - S(q)] dq$
total surplus	consumer surplus + producer surplus	$\int_0^{q^*} [D(q) - S(q)] dq$, the area between demand and supply
minimum price	a price set by the state: no unit may be sold for less	above p^* : only the quantity demanded at this price is traded, and the surpluses end there. Below p^* : no effect
average value	the constant height that gives the same area (the same total) over the interval	$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$, “average monthly ... within the first b months”: $\frac{1}{b} \int_0^b f(t) dt$
“prove that F is an antiderivative of f ”	show it by differentiating, no integrating needed	compute $F'(x)$, e.g. with the product and chain rule, and show $F'(x) = f(x)$
integration by parts	the integration rule that reverses the product rule	$\int u dv = uv - \int v du$, choose u with LIATE
mean	the sum of all values divided by their number	$\bar{x} = \frac{1}{n} \sum x_i$
median	the middle value of the sorted data	odd n : the middle value, even n : the mean of the two middle values
quartiles	Q_1 and Q_3 : the medians of the lower and the upper half of the sorted data	interquartile range $IQR = Q_3 - Q_1$, the middle 50 % of the data. With an odd n ,

Word	Meaning	Mathematics
		leave the median out of both halves
box plot	a picture of the five-number summary	a box from Q_1 to Q_3 with a line at the median, whiskers to the smallest and the largest value that are not outliers
outlier	a value far away from the rest of the data	a value outside the fences, i.e. more than $1.5 \cdot \text{IQR}$ below Q_1 or above Q_3
sample standard deviation	the typical distance of the values of a sample from their mean	$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$, for a whole population divide by N
sample space	the set of all possible outcomes of a random experiment	S , e.g. the numbers 1 to 6 for one die
event	a set of outcomes	$A \subseteq S$, with the probability $P(A)$ between 0 and 1
complement (German: <i>Gegenwahrscheinlichkeit</i>)	“not A ”, everything outside A	$P(\bar{A}) = 1 - P(A)$, e.g. “at least one” = $1 - \text{“none”}$
mutually exclusive	the events cannot happen together	$A \cap B = \emptyset$, so $P(A \cup B) = P(A) + P(B)$
independent	one event does not change the probability of the other	$P(A \cap B) = P(A) \cdot P(B)$, or $P(A B) = P(A)$, or observed count = expected count
“of the ...”, “among the ...”, “given that ...”	names the group that a percentage refers to: the condition	behind the bar: “of the members, 30 % redeem” is $P(R M) = 0.3$
tree diagram	the stages of a random experiment, with a probability on every branch	the branches that leave one point add up to 1, the second stage carries conditional probabilities
path probability	the probability of one path through the tree	multiply along the path (multiplication rule), all path probabilities add up to 1
law of total probability	the probability of an event that several paths lead to	add these paths: $P(B) = P(A) \cdot P(B A) + P(\bar{A}) \cdot P(B \bar{A})$

Word	Meaning	Mathematics
Bayes' theorem	reverses a conditional probability	$P(A B) = \frac{P(A) \cdot P(B A)}{P(B)}$, with $P(B)$ from the law of total probability
base rate (prevalence)	the share of all items (or people) that have the property, before any test	$P(D)$
true positive rate (sensitivity)	the share of the items with the property that test positive	$P(+ D)$
true negative rate (specificity)	the share of the items without the property that test negative	$P(- \bar{D})$
false positive, false negative	a positive result without the property, a negative result with the property	$\bar{D} \cap +$ and $D \cap -$, with the rates $P(+ \bar{D}) = 1 - \text{true negative rate}$ and $P(- D) = 1 - \text{true positive rate}$
"... with a positive test result does indeed ..."	among the positive results, the ones that are right	the positive predictive value $P(D +)$, the base is all positive results
"... has ... but is incorrectly identified as ..."	an item has the property, but the test says it has not, seen among all items	the joint probability $P(D \cap -) = P(D) \cdot P(- D)$, one path of the tree
positive predictive value, negative predictive value	the share of the positive results that are right, and the share of the negative results that are right	$P(D +)$ and $P(\bar{D} -)$
"Display the fractions in a contingency table. Fill in the blank fields."	write every given share into its field or total, then complete the table	the fields of a row add up to the row total, the totals add up to the grand total (1 or 100 %)
"State the businesslike meaning of the quantities ..."	say in words what a set means for the company	e.g. $\bar{M} \cap R$: "a customer who is not a member and redeems the coupon"
expected count	the count of a field if the two properties were independent	$E = \frac{\text{row total} \cdot \text{column total}}{\text{grand total}}$, compare it with the observed count
binomial distribution	the number of successes in n trials	$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$. Requirements: fixed n , two outcomes per trial, the same

Word	Meaning	Mathematics
		p in every trial, independent trials
“exactly / not more than / at least ... but not more than ...”	which values of X an event contains	$P(X = k)$, $P(X \leq k)$, $P(a \leq X \leq b)$ with both endpoints. “Fewer than” and “more than” leave out the endpoint
expected value	the long-run average of a random variable	$E[X] = \sum x_i \cdot P(X = x_i)$. Binomial: $\mu = n \cdot p$ and $\sigma = \sqrt{n \cdot p \cdot (1 - p)}$
“with a likelihood of at least ...”	the probability must reach the given value	$P(\dots) \geq$ the value, e.g. for “at least one”: $1 - (1 - p)^n \geq$ the value
minimum number (of trials)	the smallest whole number n that meets a condition	solve with logarithms, flip the sign when dividing by $\ln(1 - p) < 0$, round up and check $n - 1$ and n
interest rate per period	the rate for one compounding period	$r = \frac{\text{annual rate}}{\text{periods per year}}$, e.g. $\frac{0.036}{12}$ per month
compounded monthly / quarterly / continuously	interest is added 12 times or 4 times per year, or all the time	$A = P\left(1 + \frac{r}{n}\right)^{nt}$ with $n = 12$ or $n = 4$, continuously $A = Pe^{rt}$
nominal rate	the annual rate that the bank states, before compounding	r in $\frac{r}{n}$
effective annual rate (EAR)	the rate that gives the same growth with interest once a year	$r_{\text{eff}} = \left(1 + \frac{r}{n}\right)^n - 1$, continuously $e^r - 1$. Compare offers by their EAR
present value, discount	what a future amount is worth today, and the step from the future back to today	$PV = FV \cdot (1 + r)^{-t}$
future value	what an amount or a series of payments is worth at a later date	$FV = PV \cdot (1 + r)^t$
inflation, real interest rate	inflation: the general rise of prices. Real interest rate: the growth of what the money can buy	approximately $r_{\text{nom}} - r_{\text{infl}}$, exactly $\frac{1 + r_{\text{nom}}}{1 + r_{\text{infl}}} - 1$
annuity, instalment	equal payments at equal time intervals, and one of these payments	PMT per period, n payments, rate r per period

Word	Meaning	Mathematics
ordinary annuity (German: <i>nachschüssig</i>)	payments at the end of each period	$FV = PMT \cdot \frac{(1+r)^n - 1}{r}$, $PV = PMT \cdot \frac{1 - (1+r)^{-n}}{r}$
annuity due (German: <i>vorschüssig</i>)	payments at the beginning of each period: each payment earns one period more interest	$FV_{\text{due}} = FV \cdot (1 + r)$, $PV_{\text{due}} = PV \cdot (1 + r)$
loan, loan payment	money borrowed now and repaid with equal payments that include the interest	$PMT = PV \cdot \frac{r}{1 - (1+r)^{-n}}$, total interest = $n \cdot PMT - PV$
amortization schedule	a table with one row per payment: interest portion, principal portion and balance	interest = balance $\cdot r$, principal = $PMT - \text{interest}$, new balance = balance – principal
principal (two meanings)	(1) the amount borrowed or invested, (2) the part of a payment that repays the loan	(1) P or PV , (2) the principal portion $PMT - \text{interest}$
outstanding balance	the amount still owed after k payments	$B_k = PMT \cdot \frac{1 - (1+r)^{-(n-k)}}{r}$, the present value of the $n - k$ payments that are left
special order	a one-off extra order at a special price, made with spare capacity	with spare capacity: accept if the price lies above the short-term lower limit price (a positive contribution margin), even if it lies below the long term price floor

In set-builder notation, $\{x \in \mathbb{R} \mid \dots\}$ and $\{x \in \mathbb{R} : \dots\}$ mean the same.

Set B: Practice Mock (180 minutes, 100 points)

Problem 1: Massage Chairs for Furniture Stores [37 pts. total]

A company builds massage chairs (armchairs with a built-in massage function) and sells them to furniture stores. An analysis of its monthly costs gives the following information:

- The variable costs per unit (the variable costs divided by the output) are lowest at an output of 14 QU. There they amount to 52 MU per QU.
- At an output of 10 QU, the variable costs per unit are 60 MU per QU.
- At an output of 8 QU, the unit costs (average total cost) are 232 MU per QU.

(1 QU = 100 massage chairs, 1 MU = 1,000 EUR, so a price of p MU per QU is a price of $10p$ EUR per chair.)

- a) **Determine** the cost function assuming it is a polynomial of third degree. [12 pts.]

Hint: The fixed costs are the constant term d of C . Write $\overline{C}_v(x) = \frac{C(x)-d}{x}$ and simplify: you get a polynomial of second degree.

(for verification: $C(x) = 0.5x^3 - 14x^2 + 150x + 1296$)

- b) **Compute** the output volume with the lowest unit costs. Also, **determine** the long term price floor (the long-term lower limit price) in EUR per massage chair.

Hint: Solve the cubic equation with the polynomial solver (MENU → A → 2, degree 3), or test the whole numbers from 15 to 20.

[5 pts.]

- c) The stores buy 1,200 massage chairs per month at a price of 1,800 EUR per chair. The market is saturated at 2,100 chairs per month: the stores do not take more chairs, even for free. The demand is linear. **Determine** the price-demand function $p(x)$ with p in MU per QU, and **give** the maximum price in EUR per chair. [3 pts.]

(for verification: $p(x) = 420 - 20x$)

- d) **Show** that the monthly profit is given by $P(x) = -0.5x^3 - 6x^2 + 270x - 1296$. [2 pts.]
- e) The sales manager claims: “We already make a profit with 500 massage chairs per month, and we stay in the profit zone up to 1,400 chairs.” **Determine** the break-even point and the profit limit (upper break-even point), and **assess** both parts of the claim.

Hint: Test the whole numbers from 5 to 8 to find the break-even point.

[6 pts.]

- f) **Compute** the maximum profit and **prove** it really is a maximum. [5 pts.]
- g) **Calculate** the price per massage chair for the maximum profit. A large furniture chain wants to sell the chair under its own brand and offers half of this price, for any number of chairs. **Decide** whether the company can sign a long-term contract at this price, and whether it should deliver at this price if the contract runs only a few months. [4 pts.]

Problem 2: Curve Sketching of an Exponential Function [38 pts. total]

Let the function f be given by

$$f(x) = (x^2 - x - 4) \cdot e^{0.25x}, \quad x \in \mathbb{R}$$

Round all results to two decimal places.

- a) **State** the domain of f . **Determine** the x - and y -intercepts of G_f . [3 pts.]
- b) **Investigate** the asymptotic behavior of f . **Give** the equation of the asymptote and **state** from which side G_f approaches it. [3 pts.]
- c) **Determine** the first and the second derivative of f . [4 pts.]

(for verification: $f'(x) = 0.25(x-1)(x+8) \cdot e^{0.25x}$ and $f''(x) = \frac{1}{16}(x^2 + 15x + 20) \cdot e^{0.25x}$)

- d) **Determine** the nature and the coordinates of all stationary points. [7 pts.]
- e) **Assess** how many inflection points f has to possess at least. **Give** a reason for your answer. Then **compute** the inflection points. [4 pts.]
- f) **Determine** the tangent of f at $x = -4$. [3 pts.]
- g) **Sketch** the graph G_f in the interval $[-15; 3]$. **Draw** the tangent from f) into the same figure. [5 pts.]
- h) **Find** the antiderivative of $f(x)$ and **show** your work. [5 pts.]
- (for verification: $F(x) = (4x^2 - 36x + 128) \cdot e^{0.25x}$)
- i) G_f and the x -axis enclose a region below the x -axis. **Mark** the region in your sketch and **compute** its area. **Explain** why the integral of f from one zero to the other is not the area itself. [4 pts.]

Problem 3: Testing Blood Donations [25 pts. total]

A regional blood donation service collects 50,000 blood donations per year. Every donation is tested for the hepatitis E virus with a quick test. Donations with a positive test result (+) are discarded, donations with a negative result (−) go to the hospitals. Let V be the event “the donation contains the virus”. The prevalence (base rate) of the virus among the donations is 0.4 %. The test’s sensitivity (true positive rate) is 98 %, its specificity (true negative rate) is 99.5 %.

- a) **Create** a tree diagram for the testing and **complete** the following table for the 50,000 donations of one year. [6 pts.]

	positive (+)	negative (−)	sum
contains the virus (V)			
free of the virus ($\bar{V}$)			
sum			50,000

(for verification: 445 of the 50,000 donations test positive)

- b) **Compute** the probability that a donation is free of the virus but is incorrectly identified as infected. **Give** the number of donations per year that are discarded although they are free of the virus. [3 pts.]
- c) The head of the laboratory says: “Our test finds 98 % of the infected donations, so 98 % of the discarded donations really contain the virus.” **Compute** the probability that a donation with a positive test result does indeed contain the virus, and **assess** the statement. [4 pts.]
- d) **Determine** the number of donations which need to be tested to find at least one infected donation testing positive with a likelihood (probability) of 99 %. [4 pts.]
- e) In one week, the service tests 700 donations. **Compute** the probability for the following events:
- (1) Exactly 7 donations test positive.

- (2) Not more than 5 donations test positive.
- (3) At least 6 but not more than 10 donations test positive.
- (4) More than 12 donations test positive.

[8 pts.]

Exam Preparation Checklist

Before the exam, check that you can:

- ☐ Compute the mean, the median, the quartiles, the interquartile range and the sample standard deviation (with $n - 1$) from raw data, and find outliers with the fences (1.5 IQR below Q_1 and above Q_3)
- ☐ Read a relative frequency or a mean per day as a probability, with all items as the base
- ☐ Use the 68-95-99.7 rule, and z-scores with the values of the standard normal distribution that the task gives
- ☐ Use the complement rule (“at least one” = $1 - \text{“none”}$), the addition rule and the test $P(A \cap B) = P(A) \cdot P(B)$ for independent events, and tell mutually exclusive events from independent ones
- ☐ Count selections with the counting principle, permutations and combinations, and compute “exactly k ” and “at least one” by counting
- ☐ Draw a tree diagram with branch and path probabilities, and check that the branches at every point and all path probabilities add up to 1
- ☐ Use the law of total probability, also with an unknown rate as a linear equation (“show that ...”)
- ☐ Find the base of every conditional probability: the group after “of the ...”, “among the ...” or “given that ...” goes behind the bar
- ☐ Compute the positive and the negative predictive value with Bayes’ theorem, and tell a joint probability (“... but is incorrectly identified as ...”) from a conditional one
- ☐ Determine a maximum error rate or a minimum base rate for a required predictive value with an inequality
- ☐ Complete a contingency table from percentages (“Fill in the blank fields”) and check that every row and every column adds up to its total and the totals to the grand total
- ☐ Test independence with $P(A \cap B) = P(A) \cdot P(B)$, with $P(A | B) = P(A)$ and with the expected count, and interpret the result
- ☐ Argue whether statements are true or false, with complements such as “neither” = $\overline{A \cup B}$ and “not both” = $\overline{A \cap B}$
- ☐ Check the four requirements of the binomial distribution and compute $\mu = n \cdot p$ and $\sigma = \sqrt{n \cdot p \cdot (1 - p)}$
- ☐ Translate “exactly”, “not more than”, “at least ... but not more than ...”, “fewer than” and “more than” into values of X with the correct endpoints, and use the complement when it is shorter

- ☐ Get binomial values with the calculator: $P(X = x)$ with Binomial-Dichte and $P(X \leq x)$ with the cumulative distribution (**MENU** → **7** → **▼** → **1** → **2**), “at least” as $1 - P(X \leq k - 1)$ and “between” as $P(X \leq b) - P(X \leq a - 1)$, with the decimal comma (0,03)
- ☐ Tell the binomial from the hypergeometric distribution (a draw without replacement from a small lot), and use the geometric distribution for the first success
- ☐ Determine a minimum number n with logarithms: flip the sign when dividing by $\ln(1 - p) < 0$, round up and check $n - 1$ and n
- ☐ Use the rate per period and the number of periods in every interest and annuity formula (3.6 % compounded monthly for 5 years: $r = \frac{0.036}{12}$ and $n = 60$), and type the exact rate into the calculator
- ☐ Compare offers by their effective annual rate, and discount a future amount with $PV = FV \cdot (1 + r)^{-t}$
- ☐ Tell an ordinary annuity (payments at the end) from an annuity due (payments at the beginning), and use the annuity due factor $(1 + r)$
- ☐ Compute a loan payment, the rows of an amortization schedule, the outstanding balance and a deposit that reaches a savings target
- ☐ Round the safe way: a number of payments or periods and a required payment up, a largest affordable loan or payment down, and check the whole numbers on both sides
- ☐ Tell the unit costs $\overline{C}(x) = \frac{C(x)}{x}$ from the variable costs per unit $\overline{C}_v(x) = \frac{C(x) - C_f}{x}$, and find the output volume with the lowest unit costs and the long term price floor
- ☐ Compare a price with both lower limit prices: below the short-term lower limit price stop producing, between the two limits produce only in the short run, at or above the long term price floor all costs can be covered
- ☐ Translate cost conditions (fixed costs, “the curvature changes sign”, “the marginal costs are lowest”, total costs, variable costs per unit, unit costs) into equations and determine a cubic cost function
- ☐ Run the whole curve sketching chain for an exponential function: domain, intercepts, asymptotic behavior, stationary points, inflection points, tangent, sketch, antiderivative by integration by parts and an area

Tip

Show all work, name the base of every probability, and interpret your results in the business context.