

# FSP Training 06

Preparation for Mock Exam 07



## Instructions

This training page covers all of Section 06 and prepares you for Mock Exam 07. Set A trains setting up: you translate a text into mathematics and check your translation. Set B is a practice mock in the format of Mock Exam 07: 180 minutes plus 5 minutes reading time, 100 points, and all 3 problems must be completed. Write down every step as you would in the exam. The points show what each part is worth.

In the real FSP, decimals are written with a comma (1,25 means 1.25).

## Set A: Set Up & Check (50 minutes)

### From Text to Mathematics (xx)

**Translate** each statement into mathematics. Then answer the check question: it tells you whether your translation works. (*QU* = quantity units, *MU* = monetary units)

1. An FSP task says: “Find the antiderivative of  $f(x) = \frac{2x^4-9}{x^3}$  and show your work.” Here  $x > 0$ . **Rewrite**  $f(x)$  as a sum of powers of  $x$  and **give** all antiderivatives of  $f$ .

**Check:** **Differentiate** your antiderivative. Do you get  $f$  back? A classmate integrates the numerator and the denominator separately and gets  $G(x) = \frac{0.4x^5-9x}{0.25x^4}$ . **Compute**  $G'(1)$  and **compare** it with  $f(1)$ .

2. A company forges kitchen knives: “At an output of  $x$  QU, the marginal costs are  $C'(x) = 0.3x^2 - 2.8x + 15$  MU per QU. The total costs of 10 QU are 200 MU.” **Write down** the condition that fixes the constant of integration and **determine**  $C(x)$  and the fixed costs.

**Check:** **Compute**  $C(10)$  with your function. How much of the 200 MU are fixed costs, and how much are variable costs?

3. A new tennis club: “ $t$  weeks after the opening, new members sign up at a rate of  $n(t) = 45 + 4t$  members per week.” The club wants to know when it has 650 members. **Write** the condition “650 members have signed up” as an equation with a definite integral and **solve** it.

**Check:** **Compute**  $\int_0^T n(t) dt$  for your  $T$ . A classmate solves  $n(T) = 650$  instead and gets  $T = 151.25$ . What does the classmate’s number measure?

4. A workshop makes leather belts to order: “At an output of  $x$  belts per day ( $x \geq 10$ ), the marginal costs are  $C'(x) = 4 + \frac{80}{x}$  EUR per belt.” **Write** the rise of the daily costs when the output rises from 20 to 40 belts as a definite integral and **compute** it. Give the exact value and a decimal rounded to two decimal places.

*Check:* A classmate uses the power rule for  $\frac{80}{x} = 80x^{-1}$  and writes  $80 \cdot \frac{x^0}{0}$ . **Explain** what goes wrong. A second classmate writes  $80 \ln(40 - 20)$  for the logarithm part. **Compare** it with your result.

5. An FSP-style task says: “The graph of  $f(x) = e^{0.2x}$ , the  $x$ -axis and the lines  $x = 0$  and  $x = 10$  enclose a region. The vertical line  $x = c$  divides this region into two parts of equal size. Determine  $c$ .” **Write down** the equation for  $c$  with integrals and **solve** it. Round to two decimal places.

*Check:* A classmate takes the middle of the interval,  $c = 5$ . **Compute** the area to the left of  $x = 5$  and **compare** it with half of the whole area. Why must  $c$  lie to the right of the middle?

6. A laser tag arena in a city: “Nobody pays more than 150 EUR for a group booking (maximum price), and at a price of 0 EUR, 2,000 bookings per month would be made (market saturation). The demand is linear. The market price is 90 EUR per booking.” **Determine** the demand function  $D(q)$  for  $q$  bookings per month and the number of bookings at the market price. **Write** the consumer surplus as an integral and **compute** it.

*Check:* The region of the consumer surplus is a triangle. **Compute** its area with  $\frac{1}{2} \cdot \text{base} \cdot \text{height}$ . Do both results agree?

7. The market for honey in a region, with  $q$  in tonnes per year and prices in EUR per kg: “Demand  $D(q) = 19 - 0.25q$ , supply  $S(q) = 4 + 0.125q$ . The state sets a minimum price of 13 EUR per kg to support the beekeepers.” **Determine** the equilibrium and the quantity traded at the minimum price. **Write** the new consumer surplus and the new producer surplus as integrals and **compute** them in EUR.

*Check:* Another proposal is a minimum price of 7 EUR per kg. **Explain** why this minimum price would change nothing on the market.

8. A company starts producing a new kitchen blender: “ $t$  months after the start, the company produces  $s(t) = 50 + 7t - 0.25t^2$  hundred blenders per month.” An FSP task says: “Compute the average monthly output volume within the first 12 months.” **Write** the average as a formula with an integral and **compute** it.

*Check:* A classmate takes the mean of the first and the last value,  $\frac{s(0)+s(12)}{2}$ .

**Compute** it. Why does it differ from your result, and for which kind of function would it give the average value?

9. An FSP task says: “Find the antiderivative of  $f(x) = (2x - 3) \cdot e^{0.5x}$  and show your work.” **Write down**  $u$ ,  $du$ ,  $dv$  and  $v$  and **evaluate** the integral with integration by parts.

*Check:* **Differentiate** your result. A classmate writes  $v = 0.5e^{0.5x}$ . **Find** the classmate’s result and **show** by differentiating that it is wrong.

10. An FSP-style task says: “Determine the second derivative of  $f(x) = -0.25x^4 + 1.5x^3 + 6x^2 - 5x$ . Draw the graph of  $f''$  and use it to describe the curvature of  $G_f$ .” **Compute**  $f''(x)$  and its zeros, **sketch** the graph of  $f''$  and **explain** where  $G_f$  is concave up, where it is concave down and where its inflection points lie.

**Check:** **Compute**  $f'(0)$  and  $f''(0)$ . A classmate says: “ $f'(0) < 0$ , so  $G_f$  is concave down at  $x = 0$ .” Is the classmate right?

## One Paragraph, All Conditions (xx)

A company produces smoke detectors and sells them to DIY stores. Its marginal costs are a quadratic function of the output: they are lowest at an output of 4,000 smoke detectors per month, with 20 EUR per smoke detector. The variable costs of the first 6,000 smoke detectors per month are 138,000 EUR. The rent for the hall and the salaries of the permanent staff come to 80,000 EUR per month, even without production. The DIY stores pay 47 EUR per smoke detector, however many they buy. The company can produce at most 12,000 smoke detectors per month. The owner wants to know the output with the highest profit, a proof that it really is a maximum, and the maximum profit.

(QU = quantity units, MU = monetary units. 1 QU = 1,000 smoke detectors, 1 MU = 1,000 EUR, so a price of  $p$  MU per QU is a price of  $p$  EUR per smoke detector)

**List** every condition and every function in this paragraph as a mathematical statement, with  $x$  in QU and all money in MU. Do not solve anything.

**Check:** An assistant writes the sentence about the first 6,000 smoke detectors as  $C(6) = 138$ . **Explain** what is wrong and **give** the correct condition.

## FSP Words So Far

The FSP problems use these words from Sections 01 to 06. Learn what each one means and how to write it in mathematics.

| Word               | Meaning                                                                   | Mathematics                                                 |
|--------------------|---------------------------------------------------------------------------|-------------------------------------------------------------|
| quantity unit (QU) | a fixed number of items, e.g. 1 QU = 100 items                            | the output $x$ is counted in QU                             |
| monetary unit (MU) | a fixed amount of money, e.g. 1 MU = 100 EUR                              | results in MU, convert to EUR at the end                    |
| fixed costs        | costs that arise even if nothing is produced                              | the constant term of $C(x)$                                 |
| variable costs     | costs that grow with the output                                           | $C(x) - C(0)$ (for linear costs: output times costs per QU) |
| total costs        | fixed costs + variable costs                                              | $C(x)$                                                      |
| revenue            | quantity times price                                                      | $R(x) = x \cdot \text{price}$                               |
| profit             | revenue – total costs                                                     | $P(x) = R(x) - C(x)$                                        |
| break-even point   | the output at which the profit changes from negative (a loss) to positive | $P(x) = 0$ , the smaller zero                               |

| Word                                            | Meaning                                                                                                                | Mathematics                                                                                       |
|-------------------------------------------------|------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------|
| profit limit (upper break-even point)           | the larger output with zero profit: above it, the profit turns negative again                                          | $P(x) = 0$ , the larger zero                                                                      |
| profit zone                                     | all outputs with a positive profit                                                                                     | $P(x) > 0$ : from the break-even point to the profit limit (or the capacity)                      |
| capacity                                        | the maximum possible output                                                                                            | an upper bound: $x \leq$ capacity                                                                 |
| growth factor                                   | the factor by which a quantity is multiplied each period                                                               | $q$ in (starting value) $\cdot q^t$                                                               |
| growth rate                                     | the change per period in percent                                                                                       | $q = 1 + r$ with $r$ as a decimal, a loss of 4 % gives $q = 0.96$                                 |
| doubling time                                   | the time after which a quantity has doubled                                                                            | $q^T = 2$ , continuous: $e^{kT} = 2$ , so $T = \frac{\ln 2}{k}$                                   |
| half-life (halving time)                        | the time after which a quantity has halved                                                                             | $q^T = 0.5$ , continuous: $e^{-kT} = 0.5$ , so $T = \frac{\ln 2}{k}$                              |
| “deviates from $a$ by at most $d$ ” (tolerance) | the distance from $a$ is at most $d$                                                                                   | $ x - a  \leq d \Leftrightarrow a - d \leq x \leq a + d$                                          |
| “share of $A$ among $B$ ”                       | the part of $B$ that also belongs to $A$ : the base is $B$                                                             | $\frac{\text{number in } A \cap B}{\text{number in } B}$                                          |
| “neither $A$ nor $B$ ”                          | not in $A$ and not in $B$ , i.e. outside both circles                                                                  | $\overline{A \cup B}$                                                                             |
| contingency table                               | a table of counts for two properties: the inner fields are the regions of the Venn diagram, the margins are the totals | inner fields $A \cap B, A \cap \overline{B}, \overline{A} \cap B, \overline{A} \cap \overline{B}$ |
| gross price, net price                          | gross price = net price + VAT                                                                                          | with 19 % VAT: gross = $1.19 \cdot$ net                                                           |
| system of linear equations                      | several linear equations that must hold at the same time                                                               | solve by substitution or elimination, check the solution in every equation                        |
| no solution, infinitely many solutions          | the conditions contradict each other, or one condition only repeats another                                            | elimination leaves $0 = d$ with $d \neq 0$ , or $0 = 0$                                           |
| demand, supply                                  | the quantity buyers want to buy and the quantity sellers offer at a given price                                        | as terms in the price $p$ , e.g. $Q_d(p), Q_s(p)$ , or as prices                                  |

| Word                                 | Meaning                                                                           | Mathematics                                                                                                                                  |
|--------------------------------------|-----------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------|
|                                      |                                                                                   | $p_D(x), p_S(x)$ for the quantity $x$                                                                                                        |
| market equilibrium                   | the quantity demanded equals the quantity supplied                                | demand = supply: $Q_d(p) = Q_s(p)$ , or $p_D(x) = p_S(x)$ . With demand and supply as prices: $D(q) = S(q)$ gives $q^*$ , and $p^* = D(q^*)$ |
| price-demand function                | gives the price at which $x$ QU can be sold                                       | $p(x)$ , revenue $R(x) = x \cdot p(x)$                                                                                                       |
| average cost                         | total costs divided by the number of units                                        | $\frac{C(x)}{x}$ with $x \neq 0$                                                                                                             |
| zero form                            | all terms on one side, 0 on the other side                                        | $ax^2 + bx + c = 0$ , then factor or use the quadratic formula                                                                               |
| discriminant                         | decides how many real solutions a quadratic equation has                          | $\Delta = b^2 - 4ac$ : $> 0$ two, $= 0$ one, $< 0$ none                                                                                      |
| biquadratic equation                 | an equation with only $x^4, x^2$ and a constant                                   | substitute $u = x^2$ in $ax^4 + bx^2 + c$ , solve for $u$ , then $x = \pm\sqrt{u}$ for $u \geq 0$                                            |
| domain restrictions (of an equation) | the conditions the unknown must meet so that every term is defined                | denominators $\neq 0$ , under a root $\geq 0$ , inside a logarithm $> 0$                                                                     |
| extraneous solution                  | a candidate that solves the squared equation but not the original one             | check every candidate in the original equation                                                                                               |
| function                             | assigns to every input exactly one output                                         | $f(x)$ , its graph passes the vertical line test                                                                                             |
| domain (of a function)               | all inputs for which the function is defined, in a context only the sensible ones | e.g. $[0; \text{capacity}]$                                                                                                                  |
| range                                | all outputs that the function takes on its domain                                 | if the function grows or falls: between the values at the two ends, for a parabola also check the vertex                                     |
| maximum price                        | the price at which nobody buys any more                                           | $Q_d(p) = 0$ , or $c$ in $p = c - dQ$                                                                                                        |
| market saturation                    | the quantity demanded at a price of 0                                             | $a$ in $Q_d = a - bp$ , or the zero of $p(x)$                                                                                                |

| Word                               | Meaning                                                                                                             | Mathematics                                                                                                                        |
|------------------------------------|---------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|
| shortage, surplus                  | at a price below the equilibrium price, demand exceeds supply (shortage), above it, supply exceeds demand (surplus) | $Q_d(p) > Q_s(p)$ , or $Q_d(p) < Q_s(p)$                                                                                           |
| contribution margin                | price minus variable costs per unit: the part of the price that covers the fixed costs                              | break-even quantity = $\frac{\text{fixed costs}}{\text{contribution margin}}$                                                      |
| vertex                             | the highest or lowest point of a parabola                                                                           | $x_v = -\frac{b}{2a}$ , or $(h, k)$ in $a(x - h)^2 + k$                                                                            |
| profit-maximizing output           | the output with the highest profit                                                                                  | $x_v$ of $P(x)$ , a maximum since $a < 0$ , and the price $p(x_v)$ . If the capacity lies below $x_v$ : the capacity and its price |
| shift, stretch, compression        | moving or scaling a graph                                                                                           | $a \cdot f(b(x - h)) + k$ : first $b$ , then $h$ , then $a$ , then $k$                                                             |
| mirror image at the $y$ -axis      | the graph flipped from right to left                                                                                | $f(-x)$                                                                                                                            |
| mirror image at the $x$ -axis      | the graph flipped upside down                                                                                       | $-f(x)$                                                                                                                            |
| composite function                 | first the inner function, then the outer function                                                                   | $(f \circ g)(x) = f(g(x))$                                                                                                         |
| inverse function                   | reverses a function: gives the input that belongs to an output                                                      | solve $y = f(x)$ for $x$ , e.g. $x(p)$ from $p(x)$                                                                                 |
| polynomial of third degree         | a function with powers of $x$ up to $x^3$                                                                           | $ax^3 + bx^2 + cx + d$ with $a \neq 0$ : four unknowns, so four conditions                                                         |
| degree, leading coefficient        | the highest power of $x$ , and the number in front of it                                                            | together they decide the end behavior                                                                                              |
| zero of multiplicity $m$           | the factor $(x - c)$ appears $m$ times in the factored form                                                         | odd $m$ : the graph crosses the $x$ -axis at $c$ , even $m$ : it touches it                                                        |
| polynomial long division           | divides a polynomial by $(x - c)$ once $c$ is a known zero                                                          | $f(x) = (x - c) \cdot q(x)$ , the remainder must be 0                                                                              |
| asymptotic behavior (end behavior) | what happens to $f(x)$ for $x \rightarrow \infty$ and $x \rightarrow -\infty$                                       | polynomial: degree and sign of the leading coefficient.<br>Rational function: its asymptotes                                       |

| Word                                | Meaning                                                                            | Mathematics                                                                                                          |
|-------------------------------------|------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|
| intercepts                          | the points where the graph meets the axes                                          | $y$ -intercept $(0, f(0))$ , $x$ -intercepts from $f(x) = 0$                                                         |
| rational function                   | a quotient of two polynomials                                                      | $\frac{p(x)}{q(x)}$ with the domain $q(x) \neq 0$                                                                    |
| vertical asymptote                  | a vertical line near which the values grow without bound                           | a zero of the denominator that does not cancel                                                                       |
| hole                                | a single missing point of the graph                                                | a factor that cancels, its $y$ -value comes from the simplified term                                                 |
| horizontal asymptote                | a horizontal line that the graph approaches for $x \rightarrow \pm\infty$          | $y = c$ with $f(x) \rightarrow c$                                                                                    |
| oblique asymptote                   | a slanted line that the graph approaches for $x \rightarrow \pm\infty$             | the linear part of the long division, if the numerator's degree is one higher                                        |
| grows less than proportionally      | grows, but twice the input gives less than twice the output                        | $a \cdot x^b$ with $0 < b < 1$                                                                                       |
| continuous growth rate              | the rate $k$ of a model with the base $e$                                          | $N_0 \cdot e^{kt}$ , growth factor per period $e^k$                                                                  |
| saturation level                    | a level that a quantity approaches but never exceeds                               | a horizontal asymptote $y = L$ , e.g. of $L - a \cdot e^{-kt}$ or $\frac{L}{1 + Ae^{-kt}}$                           |
| behavior for $t \rightarrow \infty$ | the value a quantity approaches in the long run                                    | $\lim_{t \rightarrow \infty} N(t)$ , often with $e^{-kt} \rightarrow 0$                                              |
| radian                              | an angle measure in which a full turn is $2\pi$                                    | $180^\circ = \pi$ , arc length $s = r\theta$                                                                         |
| amplitude                           | half the distance between the highest and the lowest value                         | $ A  = \frac{\max - \min}{2}$                                                                                        |
| midline                             | the horizontal line halfway between the highest and the lowest value               | $y = D = \frac{\max + \min}{2}$                                                                                      |
| period                              | the length of one full cycle                                                       | $B = \frac{2\pi}{\text{period}}$                                                                                     |
| phase shift                         | the horizontal shift of a sine or cosine curve                                     | $C$ in $A \sin(B(t - C)) + D$ or $A \cos(B(t - C)) + D$                                                              |
| mirror image at the midline         | the graph flipped upside down about its midline $y = D$                            | $2D - f(t)$                                                                                                          |
| limit                               | the value that $f(x)$ approaches when $x$ approaches a number $a$ , or $\pm\infty$ | $\lim_{x \rightarrow a} f(x)$ , one-sided: $x \rightarrow a^-$ (from the left), $x \rightarrow a^+$ (from the right) |

| Word                                             | Meaning                                                                                                               | Mathematics                                                                     |
|--------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------|
| continuous (at $a$ )                             | the graph has no jump and no gap at $a$                                                                               | $f(a)$ is defined, $\lim_{x \rightarrow a} f(x)$ exists, and both are equal     |
| removable discontinuity                          | a gap that a single value could close, e.g. a hole                                                                    | the limit exists, but $f(a)$ is not defined or differs from it                  |
| derivative                                       | the instantaneous rate of change: the slope of the tangent                                                            | $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$                        |
| secant, slope of the secant                      | the line through two points of the graph, and its slope (the average rate of change)                                  | $\frac{f(b) - f(a)}{b - a}$ on $[a; b]$                                         |
| tangent, tangential gradient                     | the line that touches the graph at a point, and its slope                                                             | $y = f'(a)(x - a) + f(a)$ with the slope $f'(a)$                                |
| angle of intersection (with the $x$ -axis)       | the angle at which a line with slope $m$ crosses the $x$ -axis                                                        | $\alpha = \arctan(m)$ , with the indicator <b>D</b> for degrees                 |
| marginal cost, marginal revenue, marginal profit | about the change of the costs, the revenue or the profit caused by one more unit                                      | $C'(x), R'(x), P'(x) = R'(x) - C'(x)$                                           |
| stationary point (critical point)                | a point of the graph with a horizontal tangent                                                                        | $f'(x_0) = 0$                                                                   |
| nature of a stationary point                     | whether it is a local maximum, a local minimum or a saddle point                                                      | sign change of $f'$ at $x_0$ , or the sign of $f''(x_0)$                        |
| local maximum, local minimum                     | a point that is higher (lower) than all points nearby                                                                 | $f'(x_0) = 0$ and $f''(x_0) < 0$ (maximum) or $f''(x_0) > 0$ (minimum)          |
| saddle point                                     | a stationary point at which $f'$ does not change its sign: the graph levels off and keeps its direction               | $f'(x_0) = 0$ without a sign change, e.g. $f''(x_0) = 0$ and $f'''(x_0) \neq 0$ |
| necessary condition, sufficient condition        | a condition every candidate must meet, and a condition that proves the kind of point ("prove it really is a maximum") | necessary: $f'(x_0) = 0$ , sufficient: e.g. $f''(x_0) < 0$ for a maximum        |
| global maximum, global minimum (on an interval)  | the highest or lowest value on the whole interval                                                                     | compare the local extrema with the values at the ends of the interval           |
| concave up, concave down                         | the graph curves upward like a cup, or downward like a cap                                                            | $f''(x) > 0$ , or $f''(x) < 0$                                                  |

| Word                                             | Meaning                                                                                                | Mathematics                                                                                           |
|--------------------------------------------------|--------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------|
| inflection point                                 | the point at which the curvature changes sign: the graph is steepest or flattest there                 | $f''(x_0) = 0$ and $f'''(x_0) \neq 0$ , or a sign change of $f''$                                     |
| “the curvature changes sign”                     | the graph changes from concave down to concave up, or back                                             | an inflection point: $f''(x_0) = 0$ . For a cubic cost function, the marginal costs are lowest there  |
| inflectional tangent                             | the tangent at an inflection point: it shows the fastest growth or decline                             | $y = f'(a)(x - a) + f(a)$ with $f''(a) = 0$                                                           |
| unit costs, lowest unit costs                    | the total costs per unit, and the output at which they are lowest                                      | $k(x) = \frac{C(x)}{x}$ , minimum from $k'(x) = 0$ and $k''(x) > 0$                                   |
| long-term price floor                            | the lowest unit costs: below this price, the company cannot cover its total costs in the long run      | the minimum value of $k(x) = \frac{C(x)}{x}$                                                          |
| variable costs per unit                          | the variable costs divided by the output                                                               | $k_v(x) = \frac{C(x) - C(0)}{x}$                                                                      |
| short-term lower limit price                     | the lowest variable costs per unit: below this price, the company cannot even cover its variable costs | the minimum value of $k_v(x)$                                                                         |
| function family (German: <i>Funktionsschar</i> ) | functions with the same term that differ only in a parameter                                           | $f_a(x)$ , e.g. $f_a(x) = x^2 - ax + 1$                                                               |
| parameter                                        | a letter that is fixed within one function of the family but changes from one function to the next     | solve a condition for it, e.g. $f'_a(2) = 0$                                                          |
| antiderivative (German: <i>Stammfunktion</i> )   | a function whose derivative is $f$                                                                     | $F'(x) = f(x)$                                                                                        |
| constant of integration                          | the free constant: two antiderivatives of $f$ differ only by a constant                                | $F(x) + C$ , fixed by one condition, e.g. $C(0) = \text{fixed costs}$ or $P(0) = -\text{fixed costs}$ |
| indefinite integral                              | the family of all antiderivatives                                                                      | $\int f(x) dx = F(x) + C$                                                                             |
| definite integral                                | a number: the signed area between the graph and the $x$ -axis from $a$ to $b$                          | $\int_a^b f(x) dx = F(b) - F(a)$ (Fundamental Theorem of Calculus)                                    |

| Word                                          | Meaning                                                                         | Mathematics                                                                                                                          |
|-----------------------------------------------|---------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------|
| lower limit, upper limit                      | the start $a$ and the end $b$ of the interval in $\int_a^b f(x) dx$             | “in the second month”: from $t = 1$ to $t = 2$                                                                                       |
| signed area                                   | area above the $x$ -axis counts positive, area below it negative                | $\int_a^b f(x) dx$ , it can be negative or 0                                                                                         |
| area (between a graph and the $x$ -axis)      | always a positive number                                                        | split at the zeros and add the absolute values of the integrals of the pieces                                                        |
| net change                                    | the total change of a quantity, computed from its rate of change                | $F(b) - F(a) = \int_a^b F'(x) dx$ ,<br>e.g. extra costs $\int_a^b C'(x) dx$                                                          |
| “total ... within the first ... months”       | everything that has come together from the start up to month $b$                | $\int_0^b (\text{rate}) dt$                                                                                                          |
| area enclosed by two graphs                   | the region between the graphs from one intersection point to the next           | $\int_a^b [f(x) - g(x)] dx$ with $f$ on top, limits from $f(x) = g(x)$ , split where the graphs cross                                |
| “both regions have equal size”                | two regions have the same area (not: the integral over both is 0)               | compute both areas separately and compare them                                                                                       |
| consumer surplus                              | what the buyers gain: they would have paid more than the market price           | $CS = \int_0^{q^*} [D(q) - p^*] dq$                                                                                                  |
| producer surplus                              | what the sellers gain: they would have sold for less than the market price      | $PS = \int_0^{q^*} [p^* - S(q)] dq$                                                                                                  |
| total surplus                                 | consumer surplus + producer surplus                                             | $\int_0^{q^*} [D(q) - S(q)] dq$ , the area between demand and supply                                                                 |
| minimum price                                 | a price set by the state: no unit may be sold for less                          | above $p^*$ : only the quantity demanded at this price is traded, and the surpluses end there. Below $p^*$ : no effect               |
| average value                                 | the constant height that gives the same area (the same total) over the interval | $f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$ ,<br>“average monthly ... within the first $b$ months”:<br>$\frac{1}{b} \int_0^b f(t) dt$ |
| “prove that $F$ is an antiderivative of $f$ ” | show it by differentiating, no integrating needed                               | compute $F'(x)$ , e.g. with the product and chain rule, and show $F'(x) = f(x)$                                                      |

| Word                 | Meaning                                             | Mathematics                                                |
|----------------------|-----------------------------------------------------|------------------------------------------------------------|
| integration by parts | the integration rule that reverses the product rule | $\int u \, dv = uv - \int v \, du$ , choose $u$ with LIATE |

In set-builder notation,  $\{x \in \mathbb{R} \mid \dots\}$  and  $\{x \in \mathbb{R} : \dots\}$  mean the same.

Set B: Practice Mock (180 minutes, 100 points)

Problem 1: A Cubic Read from Its Graph [25 pts. total]

The figure shows the graph  $G_f$  of a function  $f$  on the interval  $[-5.5; 2.5]$ . The graph touches the  $x$ -axis at  $x = 1$ . The three marked points lie on grid points. Round to two decimal places.

*Note: Print the page or copy the figure onto squared paper. In the exam, you draw directly into the printed figure.*

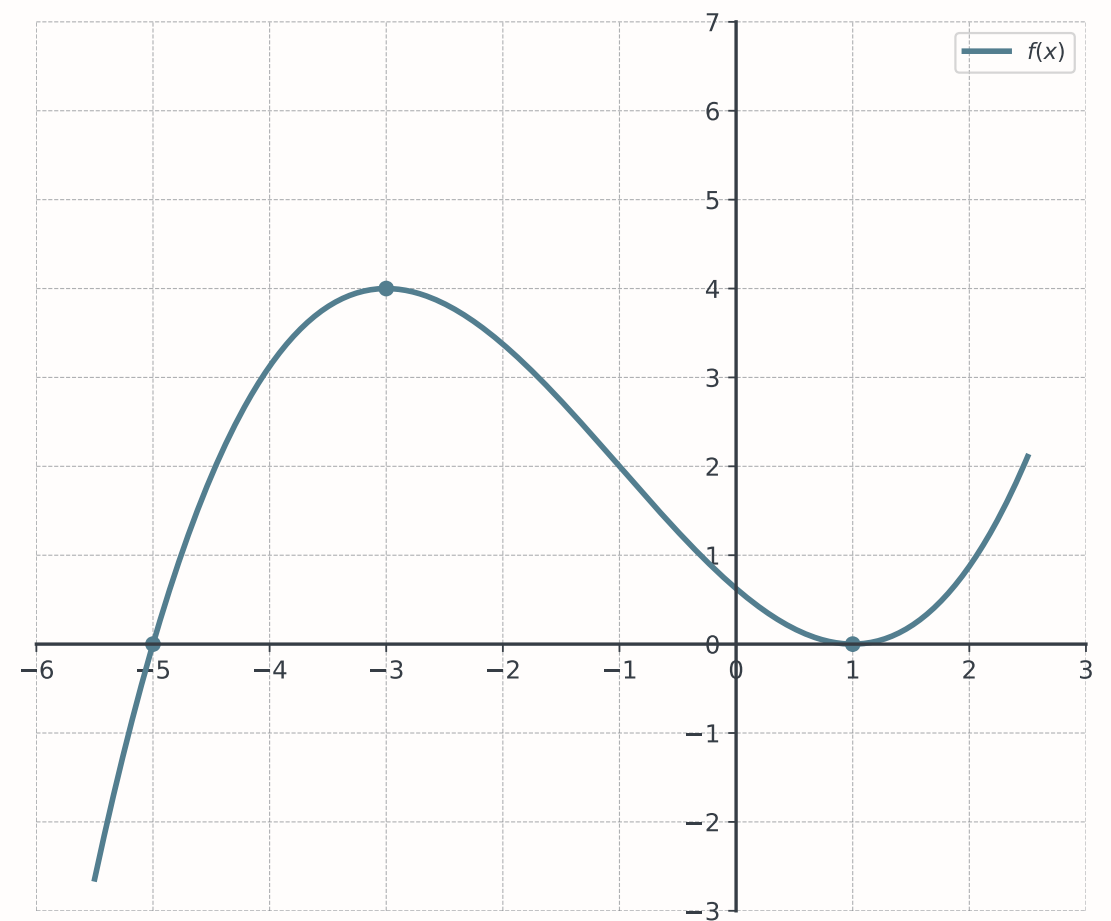


Figure 1: Graph of  $f$

- a) **Graph** the derivative  $f'$  directly into the figure. **Give** the zeros of  $f'$  and the  $x$ -value at which  $f'$  has its lowest value, and **explain** each of them with the graph of  $f$ . [5 pts.]
- b) **Determine** the  $x$ -values of the local extrema and of the inflection points of an antiderivative  $F$  of  $f$ , with the type of each extremum. **Give** the intervals on which  $F$  is concave up and concave down. Then **sketch** the antiderivative  $F$  whose graph passes through the point  $(1, 0)$  in a new coordinate system.

*Hint: Because  $F' = f$ , the rules from Session 05-05 apply to  $F$ :  $F$  increases where  $f$  is positive,  $F$  has a local extremum where  $f$  changes sign, and  $F$  has an inflection point where  $f$  has a local extremum.*

[7 pts.]

- c)  $G_f$  is the graph of a polynomial function of third degree,  $f(x) = ax^3 + bx^2 + cx + d$ . **Read** four conditions from the figure and **determine**  $a$ ,  $b$ ,  $c$  and  $d$  computationally. [6 pts.]

(for verification:  $f(x) = 0.125x^3 + 0.375x^2 - 1.125x + 0.625$ )

- d) **Determine** the equation of the tangent  $t$  to  $G_f$  at  $x = -4$ . **Determine** the other point of  $G_f$  at which the tangent is parallel to  $t$ . [5 pts.]
- e) A line with slope  $m$  meets the  $x$ -axis at the angle  $\alpha = \arctan(m)$ . **Compute** the angle of intersection  $\alpha$  between  $t$  and the  $x$ -axis.

*Hint: The indicator **D** must show. **SHIFT** + **tan** gives  $\arctan$ , just as **SHIFT** + **sin** gives  $\sin^{-1}$ .*

[2 pts.]

## Problem 2: An Exponential Function, Its Antiderivatives and an Area [40 pts. total]

Let the function  $f$  be given by

$$f(x) = (4 - x^2) \cdot e^{0.5x}, \quad x \in \mathbb{R}$$

Round all results to two decimal places.

- a) **Determine** the  $x$ - and  $y$ -intercepts of  $G_f$ . **Investigate** the asymptotic behavior of  $f$ .

*Hint: For  $x \rightarrow -\infty$ , put  $t = -x$ . Then  $e^{0.5x} = e^{-0.5t}$  with  $t \rightarrow \infty$ .*

[4 pts.]

- b) **Determine** the first and the second derivative of  $f$ . **Factor out**  $e^{0.5x}$  in both. [5 pts.]

(for verification:  $f'(x) = -0.5(x^2 + 4x - 4) \cdot e^{0.5x}$  and  $f''(x) = -0.25(x^2 + 8x + 4) \cdot e^{0.5x}$ )

- c) **Determine** the nature and the coordinates of all stationary points of  $f$ . [5 pts.]
- d) **Determine** the inflection points of  $f$ . [4 pts.]
- e) **Sketch**  $G_f$  in the interval  $[-8; 2.5]$ . **Mark** the intercepts, the stationary points and the inflection points. [5 pts.]

- f) **Show** that  $G_f$  is not point-symmetric to the origin (that would mean  $f(-x) = -f(x)$  for every  $x$ ). **Give** a function  $f_1(x)$  which is a mirror image of  $f(x)$  mirrored at the  $y$ -axis and **give** a reason for your choice. [3 pts.]
- g) **Prove** that  $F$  given by  $F(x) = -2(x-2)^2 \cdot e^{0.5x}$  is an antiderivative of  $f$ . [4 pts.]
- h) **Give** all antiderivatives of  $f$ . **Determine** the antiderivative  $F_0$  of  $f$  whose graph passes through the point  $(-2, 0)$ . [3 pts.]
- (for verification:  $F_0(x) = -2(x-2)^2 \cdot e^{0.5x} + 32e^{-1}$ )
- i) Between its two zeros,  $G_f$  and the  $x$ -axis enclose a region. **Compute** its area. [3 pts.]
- j) The vertical line  $x = c$  with  $-2 < c < 2$  cuts the region from i) into a left and a right part with the same area. **Show** that  $c$  satisfies  $(c-2)^2 \cdot e^{0.5c} = 8e^{-1}$ , and **solve** this equation. **Explain** with f) why  $c$  lies to the right of the  $y$ -axis.

*Hint: Solve the last equation with SOLVE (SHIFT + CALC) and the start value 0.*

[4 pts.]

### Problem 3: Grapes and Wine from a Wine Region [35 pts. total]

In a wine region, winegrowers sell their grapes to the wineries of the region. The market for grapes is modelled with  $q$  in thousand tonnes of grapes per year and prices in EUR per tonne. The winegrowers would sell the  $q$ -th thousand tonnes for at least

$$S(q) = 290 + 7q + 0.5q^2$$

EUR per tonne (supply function). Round to two decimal places.

- a) **Determine** the linear demand function assuming a maximum price of 1,700 EUR per tonne and a market saturation at 68,000 tonnes of grapes per year. **Give**  $D(q)$  with  $q$  in thousand tonnes. [3 pts.]

(for verification:  $D(q) = 1700 - 25q$ )

- b) **Determine** the market equilibrium. [4 pts.]
- c) **Compute** the consumer surplus and the producer surplus in EUR. **Explain** what the producer surplus means for the winegrowers. [7 pts.]
- d) After a year with low prices, the state guarantees a minimum price of 1,050 EUR per tonne: no grapes may be sold for less. **Determine** the quantity that the wineries buy and the quantity that the winegrowers want to sell at this price. The state buys the remaining grapes at the minimum price. **Compute** the costs for the state and the new consumer surplus of the wineries. [6 pts.]

One winery of the region makes wine from its own grapes and sells it in bottles. At an output of  $x$  thousand bottles per year, its marginal profit is

$$P'(x) = -0.06x^2 + 1.56x + 7.2$$

EUR per bottle. The fixed costs of the winery are 250,000 EUR per year.

- e) **Determine** the output with the highest profit and **prove** that no other output brings a higher profit. [4 pts.]

- f) **Determine** the profit function  $P(x)$  in thousand EUR. **Compute** the maximum profit in EUR. [4 pts.]

A sparkling wine cellar of the region launches a new sparkling wine.  $t$  months after the launch, it sells

$$s(t) = 8 - 6e^{-t/3}$$

thousand bottles per month.

- g) **Compute** the average monthly sales volume within the first 12 months. **Determine** the time at which the monthly sales reach this average. [7 pts.]

## Exam Preparation Checklist

Before the exam, ensure you can:

- ☐ Compute limits from a graph, by factoring and cancelling and for  $x \rightarrow \pm\infty$ , and check continuity with the three conditions
- ☐ Tell a hole (removable discontinuity) from a vertical asymptote with one-sided limits
- ☐ Compute the slope of a secant and the tangential gradient, and interpret marginal cost, marginal revenue and marginal profit
- ☐ Differentiate with the power, product, quotient and chain rules, including  $e^x$ ,  $e^{g(x)}$  and  $\ln(x)$ , and check a derivative value with the d/dx template
- ☐ Investigate the asymptotic behavior of functions like  $(x^2 + bx + c) \cdot e^{-kx}$  with  $x^n \cdot e^{-kx} \rightarrow 0$  for  $x \rightarrow \infty$
- ☐ Determine the nature and the coordinates of all stationary points with the sign change of  $f'$  or the second derivative test, and recognize a saddle point
- ☐ Determine inflection points with  $f''$  (sign change or  $f''' = 0$ ), assess how many inflection points a function has to possess at least, and explain their meaning, e.g. the fastest growth or the lowest marginal costs
- ☐ Determine a tangent or an inflectional tangent and the angle of intersection  $\alpha = \arctan(m)$  with the indicator **D**
- ☐ Find global extrema on a closed interval with the values at the ends, e.g. the highest profit within the capacity
- ☐ Sketch a graph with the 6-step algorithm and argue whether statements about  $f$ ,  $f'$  and  $f''$  are true or false
- ☐ Determine a cubic cost, revenue or sales function from conditions: an extremum or a tangent gives two equations, an inflectional tangent three
- ☐ Get the cost function from the marginal costs and the fixed costs
- ☐ Compute the maximum profit, prove that it is a maximum and give the price, and determine the break-even point and the profit limit
- ☐ Compute the lowest unit costs, the long-term price floor and the short-term lower limit price, and decide with them
- ☐ Work with a function family: stationary points in terms of the parameter, and the parameter for a given location or value

- ☐ Graph  $f'$  from the graph of  $f$ , and sketch a possible antiderivative  $F$  from the graph of  $f$ :  $F$  increases where  $f$  is positive, has a local extremum where  $f$  changes sign and an inflection point where  $f$  has a local extremum
- ☐ Find antiderivatives with the power, constant multiple and sum rules,  $\int e^{ax} dx$  and  $\int \frac{1}{x} dx$ , and fix the constant of integration with a condition such as  $C(0) = \text{fixed costs}$  or  $P(0) = -\text{fixed costs}$
- ☐ Prove that  $F$  is an antiderivative of  $f$  by differentiating, and find antiderivatives of products such as  $(ax + b) \cdot e^{kx}$  with integration by parts
- ☐ Compute definite integrals with the Fundamental Theorem of Calculus, and tell the signed area from the area by splitting at the zeros
- ☐ Compute totals and net changes from rates, e.g. “the total ... within the first ... months” or the extra costs  $\int_a^b C'(x) dx$ , and solve for an unknown limit
- ☐ Compute the area enclosed by two graphs, set up the equation for a line that divides a region into two parts of equal size, and show computationally that two regions have equal size
- ☐ Determine the market equilibrium, the consumer surplus and the producer surplus, and what a minimum price changes
- ☐ Compute the average value of a function, e.g. the average monthly sales volume within the first ... months

 **Tip**

Remember: Show all work, label units, and interpret your results in the business context.