Mock Exam 04-07 Preparation

Use this to practice for the mock exam.

Exam Preparation Tasks

These tasks are designed to prepare you for Mini-Mock Exam 04. Work through each problem systematically, showing all steps as you would in the actual exam.

Task 1: Exponential Growth Practice [x]

A technology startup's user base grows exponentially. Initially, they have 1,000 users. After 2 months, they have 9,000 users.

a) Find the exponential growth function $U(t) = U_0 \cdot a^t$ where t is time in months.

b) Calculate the number of users after 3 months.

c) When will the user base reach 81,000 users?

Solution to Task 1

Part a: Find growth function

Step 1: Set up known information - $U(0) = 1000$ (initial) - $U(2) = 9000$ (after 2 months)

Step 2: Use the condition at t = 2

$$9000 = 1000 \cdot a^2$$

$$a^2 = 9$$

$$a = 3$$

(taking positive root)

Step 3: Write the function

$$U(t) = 1000 \cdot 3^t$$

Part b: Users after 3 months

$$U(3) = 1000 \cdot 3^3 = 1000 \cdot 27 = 27,000$$

Answer: 27,000 users

Part c: When reaching 81,000 users

Step 1: Set up equation

$$1000 \cdot 3^t = 81000$$

$$3^t = 81$$

Step 2: Recognize that $81 = 3^4$

$$3^t = 3^4$$

$$t = 4$$

Answer: The user base will reach 81,000 users after 4 months

Task 2: Transformation Analysis [xx]

Consider the function $f(x) = \ln(x)$ and its transformation $g(x) = -3\ln(x-2) + 5$.

a) List all transformations applied to obtain $g(x)$ from $f(x)$.

b) Determine the domain and range of $g(x)$.

c) Find where $g(x) = 2$ algebraically.

d) Sketch both $f(x)$ and $g(x)$ on the same axes for $x \in [0.1, 10]$.

Solution to Task 2

Part a: List transformations

1. Horizontal shift right 2 units: $\ln(x) \to \ln(x-2)$
2. Vertical stretch by factor 3: $\ln(x-2) \to 3\ln(x-2)$
3. Reflection over x-axis: $3\ln(x-2) \to -3\ln(x-2)$
4. Vertical shift up 5 units: $-3\ln(x-2) \to -3\ln(x-2)+5$

Part b: Domain and range

Domain: - Need $x-2>0$ - Therefore $x>2$ - Domain: $(2,\infty)$

Range: - As $x \to 2^+$: $g(x) \to -3(-\infty)+5 = +\infty$ - As $x \to \infty$: $g(x) \to -3(+\infty)+5 = -\infty$ - Range: $(-\infty,\infty)$

Part c: Solve $g(x)=2$

Step 1: Set up equation

$$-3\ln(x-2)+5=2$$

Step 2: Isolate logarithm

$$-3\ln(x-2)=-3$$

$$\ln(x-2)=1$$

Step 3: Convert to exponential form

$$x-2=e^1=e$$

$$x=2+e\approx 4.718$$

Part d: Sketch

Key points for $g(x)$: - Vertical asymptote at $x=2$ - Point $(3,5-3\ln(1))=(3,5)$ - Point $(2+e,2)$ from part c - Decreasing function

Task 3: Combined Exponential and Logarithmic [xx]

A chemical reaction follows the model $C(t)=64\cdot 2^{-t/3}$ where $C(t)$ is concentration in mg/L and t is time in hours.

a) What is the initial concentration?

b) Find the half-life (time for concentration to reduce by half).

c) After how many hours will the concentration be 2 mg/L?

d) Express the time t as a function of concentration C.

ℹ Solution to Task 3

Part a: Initial concentration

$$C(0) = 64 \cdot 2^0 = 64 \quad \text{mg/L}$$

Part b: Half-life

Step 1: Set up equation for half of initial concentration

$$32 = 64 \cdot 2^{-t/3}$$

Step 2: Simplify

$$\frac{1}{2} = 2^{-t/3}$$

$$2^{-1} = 2^{-t/3}$$

Step 3: Equate exponents

$$-1 = -\frac{t}{3}$$

$$t = 3$$

Answer: Half-life is 3 hours

Part c: When concentration is 2 mg/L

Step 1: Set up equation

$$2 = 64 \cdot 2^{-t/3}$$

Step 2: Simplify

$$\frac{2}{64} = 2^{-t/3}$$

$$\frac{1}{32} = 2^{-t/3}$$

$$2^{-5} = 2^{-t/3}$$

Step 3: Equate exponents

$$-5 = -\frac{t}{3}$$

$$t = 15$$

Answer: After 15 hours

Part d: Express t as function of C

Step 1: Start with $C = 64 \cdot 2^{-t/3}$

Step 2: Isolate the exponential term

$$\frac{C}{64} = 2^{-t/3}$$

Step 3: Take logarithm base 2

$$\log_2\left(\frac{C}{64}\right) = -\frac{t}{3}$$

Task 4: Rational Function Analysis [xxx]

Analyze the function $f(x) = \frac{x^2-9}{x^2-4x+3}$.

a) Factor completely and identify any holes.

b) Find all vertical asymptotes.

c) Determine the horizontal asymptote.

d) Find all intercepts.

e) Sketch the function, clearly marking all features.

> **i Solution to Task 4**
>
> Part a: Factor and find holes
> Step 1: Factor numerator and denominator - Numerator: $x^2 - 9 = (x-3)(x+3)$ - Denominator: $x^2 - 4x + 3 = (x-3)(x-1)$
>
> Step 2: Identify common factors
>
> $$f(x) = \frac{(x-3)(x+3)}{(x-3)(x-1)}$$
>
> Common factor: $(x-3)$ creates a hole at $x = 3$
>
> Step 3: Find hole coordinates Simplified form: $f(x) = \frac{x+3}{x-1}$ for $x \neq 3$ At $x = 3$: $y = \frac{3+3}{3-1} = \frac{6}{2} = 3$ Hole at $(3, 3)$
>
> Part b: Vertical asymptotes
> From simplified form: $x - 1 = 0$ gives $x = 1$ Vertical asymptote at $x = 1$
>
> Part c: Horizontal asymptote
> Original function has equal degrees (both 2) Leading coefficients: numerator = 1, denominator = 1 Horizontal asymptote: $y = 1$
>
> Part d: Intercepts
> x-intercepts: Set numerator = 0 in simplified form $x + 3 = 0$ gives $x = -3$
>
> y-intercept: $f(0) = \frac{0+3}{0-1} = -3$
>
> Intercepts: $(-3, 0)$ and $(0, -3)$
>
> Part e: Key features for sketch
> - Vertical asymptote: $x = 1$ (dashed line)
> - Horizontal asymptote: $y = 1$ (dashed line)
> - Hole: $(3, 3)$ (open circle)
> - Intercepts: $(-3, 0)$ and $(0, -3)$

Task 5: Trigonometric Modeling [xxx]

The height of a Ferris wheel car above ground is modeled by:

$$h(t) = 25 - 20\cos\left(\frac{\pi t}{30}\right)$$

where h is in meters and t is time in seconds.

a) Find the amplitude, period, and vertical shift.

b) What are the maximum and minimum heights?

c) At what times in the first minute is the car at exactly 25 meters?

d) How long does it take to go from minimum to maximum height?

Solution to Task 5

Part a: Amplitude, period, vertical shift

- Amplitude: $|-20| = 20$ meters
- Period: $\frac{2\pi}{\pi/30} = \frac{2\pi \cdot 30}{\pi} = 60$ seconds
- Vertical shift: 25 meters (upward)

Part b: Maximum and minimum heights

- Maximum: When $\cos = -1$: $h = 25 - 20(-1) = 45$ meters
- Minimum: When $\cos = 1$: $h = 25 - 20(1) = 5$ meters

Part c: When height is 25 meters

Step 1: Set up equation

$$25 - 20\cos\left(\frac{\pi t}{30}\right) = 25$$

$$-20\cos\left(\frac{\pi t}{30}\right) = 0$$

$$\cos\left(\frac{\pi t}{30}\right) = 0$$

Step 2: Solve for t

$$\frac{\pi t}{30} = \frac{\pi}{2} + n\pi$$

$$t = 15 + 30n$$

Step 3: Find times in first minute - $n = 0$: $t = 15$ seconds - $n = 1$: $t = 45$ seconds

Answer: At 15 seconds and 45 seconds

Part d: Time from minimum to maximum

The minimum occurs when $\cos = 1$, maximum when $\cos = -1$. This is half a period: $\frac{60}{2} = 30$ seconds

Task 6: Logarithmic Applications [xxx]

The pH of a solution is given by $pH = -\log_{10}[H^+]$ where $[H^+]$ is hydrogen ion concentration in moles per liter.

a) If a solution has pH = 3.5, find the hydrogen ion concentration.

b) If the concentration doubles, by how much does the pH change?

c) What concentration gives a neutral pH of 7?

d) Express $[H^+]$ as a function of pH.

Solution to Task 6

Part a: Find concentration at pH = 3.5
Step 1: Use the pH formula

$$3.5 = -\log_{10}[H^+]$$

$$-3.5 = \log_{10}[H^+]$$

Step 2: Convert to exponential form

$$[H^+] = 10^{-3.5} = \frac{1}{10^{3.5}} = \frac{1}{\sqrt{10^7}} = \frac{1}{\sqrt{10,000,000}}$$

$$[H^+] = 10^{-3.5} \approx 3.16 \times 10^{-4}$$

moles/L

Part b: pH change when concentration doubles
Step 1: Original pH

$$pH_1 = -\log_{10}[H^+]$$

Step 2: New pH with doubled concentration

$$pH_2 = -\log_{10}(2[H^+]) = -\log_{10}(2) - \log_{10}[H^+]$$

$$pH_2 = -\log_{10}(2) + pH_1$$

Step 3: Change in pH

$$\Delta pH = pH_2 - pH_1 = -\log_{10}(2) \approx -0.301$$

Answer: pH decreases by approximately 0.301 units

Part c: Concentration for neutral pH

$$7 = -\log_{10}[H^+]$$

$$[H^+] = 10^{-7}$$

moles/L

Part d: Express concentration as function of pH
Step 1: Start with $pH = -\log_{10}[H^+]$

Step 2: Solve for $[H^+]$

$$-pH = \log_{10}[H^+]$$

$$[H^+] = 10^{-pH}$$

Task 7: Function Comparison [xxxx]

Consider three investment strategies with values after t years: - Strategy A: $A(t) = 10000(1.05)^t$ (5% annual compound interest) - Strategy B: $B(t) = 10000 + 600t$ (linear growth) - Strategy C: $C(t) = 10000\sqrt{1 + 0.1t}$ (square root growth)

a) Which strategy has the highest value after 10 years?

b) Find when Strategy A overtakes Strategy B.

c) Prove that Strategy A eventually exceeds both other strategies for large t.

d) Find the average rate of change for each strategy over the first 5 years.

Solution to Task 7

Part a: Values after 10 years

- $A(10) = 10000(1.05)^{10} = 10000(1.6289) = 16,289$
- $B(10) = 10000 + 600(10) = 16,000$
- $C(10) = 10000\sqrt{1+1} = 10000\sqrt{2} \approx 14,142$

Answer: Strategy A has the highest value at $16,289

Part b: When A overtakes B

Step 1: Set up equation

$$10000(1.05)^t = 10000 + 600t$$

$$(1.05)^t = 1 + 0.06t$$

Step 2: Test values - At $t = 15$: $(1.05)^{15} = 2.079$ vs $1 + 0.9 = 1.9 \longrightarrow$ A wins - At $t = 14$: $(1.05)^{14} = 1.980$ vs $1 + 0.84 = 1.84 \longrightarrow$ A wins - At $t = 13$: $(1.05)^{13} = 1.886$ vs $1 + 0.78 = 1.78 \longrightarrow$ A wins - At $t = 12$: $(1.05)^{12} = 1.796$ vs $1 + 0.72 = 1.72 \longrightarrow$ A wins - At $t = 11$: $(1.05)^{11} = 1.710$ vs $1 + 0.66 = 1.66 \longrightarrow$ A wins - At $t = 10$: $(1.05)^{10} = 1.629$ vs $1 + 0.6 = 1.6 \longrightarrow$ A wins

More precisely, A overtakes B between year 9 and 10.

Part c: Long-term dominance of A

Proof: - $\lim_{t\to\infty} \frac{A(t)}{B(t)} = \lim_{t\to\infty} \frac{10000(1.05)^t}{10000+600t} = \infty$ (exponential beats linear) - $\lim_{t\to\infty} \frac{A(t)}{C(t)} = \lim_{t\to\infty} \frac{10000(1.05)^t}{10000\sqrt{1+0.1t}} = \infty$ (exponential beats square root)

Therefore, Strategy A eventually dominates both others.

Part d: Average rates of change over first 5 years

Formula: Average rate = $\frac{f(5)-f(0)}{5-0}$

Strategy A:

$$\frac{A(5) - A(0)}{5} = \frac{10000(1.05)^5 - 10000}{5} = \frac{10000(1.2763 - 1)}{5} = 552.60$$

Strategy B:

$$\frac{B(5) - B(0)}{5} = \frac{13000 - 10000}{5} = 600$$

Strategy C:

$$\frac{C(5) - C(0)}{5} = \frac{10000\sqrt{1.5} - 10000}{5} = \frac{10000(1.2247 - 1)}{5} = 449.40$$

Answer: Average rates: A = $552.60/year, B = $600/year, C = $449.40/year

Exam Readiness Checklist

After completing these tasks, you should be able to:

- [?] Model exponential growth and decay
- [?] Apply function transformations systematically
- [?] Solve logarithmic equations
- [?] Analyze rational functions completely
- [?] Work with trigonometric models
- [?] Solve optimization problems
- [?] Compare different function types