

Mock Exam 04-07 Preparation

Use this to practice for the mock exam.

Exam Preparation Tasks

These tasks are designed to prepare you for Mini-Mock Exam 04. Work through each problem systematically, showing all steps as you would in the actual exam.

Task 1: Exponential Growth Practice [x]

A technology startup's user base grows exponentially. Initially, they have 1,000 users. After 2 months, they have 9,000 users.

- a) Find the exponential growth function $U(t) = U_0 \cdot a^t$ where t is time in months.
- b) Calculate the number of users after 3 months.
- c) When will the user base reach 81,000 users?

i Solution to Task 1

Part a: Find growth function

Step 1: Set up known information - $U(0) = 1000$ (initial) - $U(2) = 9000$ (after 2 months)

Step 2: Use the condition at $t = 2$

$$9000 = 1000 \cdot a^2$$

$$a^2 = 9$$

$$a = 3$$

(taking positive root)

Step 3: Write the function

$$U(t) = 1000 \cdot 3^t$$

Part b: Users after 3 months

$$U(3) = 1000 \cdot 3^3 = 1000 \cdot 27 = 27,000$$

Answer: 27,000 users

Part c: When reaching 81,000 users

Step 1: Set up equation

$$1000 \cdot 3^t = 81000$$

$$3^t = 81$$

Step 2: Recognize that $81 = 3^4$

$$3^t = 3^4$$

$$t = 4$$

Answer: The user base will reach 81,000 users after 4 months

Task 2: Transformation Analysis [xx]

Consider the function $f(x) = \ln(x)$ and its transformation $g(x) = -3 \ln(x - 2) + 5$.

- List all transformations applied to obtain $g(x)$ from $f(x)$.
- Determine the domain and range of $g(x)$.
- Find where $g(x) = 2$ algebraically.
- Sketch both $f(x)$ and $g(x)$ on the same axes for $x \in [0.1, 10]$.

i Solution to Task 2

Part a: List transformations

1. Horizontal shift right 2 units: $\ln(x) \rightarrow \ln(x - 2)$
2. Vertical stretch by factor 3: $\ln(x - 2) \rightarrow 3 \ln(x - 2)$
3. Reflection over x-axis: $3 \ln(x - 2) \rightarrow -3 \ln(x - 2)$
4. Vertical shift up 5 units: $-3 \ln(x - 2) \rightarrow -3 \ln(x - 2) + 5$

Part b: Domain and range

Domain: - Need $x - 2 > 0$ - Therefore $x > 2$ - Domain: $(2, \infty)$

Range: - As $x \rightarrow 2^+$: $g(x) \rightarrow -3(-\infty) + 5 = +\infty$ - As $x \rightarrow \infty$: $g(x) \rightarrow -3(+\infty) + 5 = -\infty$ - Range: $(-\infty, \infty)$

Part c: Solve $g(x) = 2$

Step 1: Set up equation

$$-3 \ln(x - 2) + 5 = 2$$

Step 2: Isolate logarithm

$$-3 \ln(x - 2) = -3$$

$$\ln(x - 2) = 1$$

Step 3: Convert to exponential form

$$x - 2 = e^1 = e$$

$$x = 2 + e \approx 4.718$$

Part d: Sketch

Key points for $g(x)$: - Vertical asymptote at $x = 2$ - Point $(3, 5 - 3 \ln(1)) = (3, 5)$ - Point $(2 + e, 2)$ from part c - Decreasing function

Task 3: Combined Exponential and Logarithmic [xx]

A chemical reaction follows the model $C(t) = 64 \cdot 2^{-t/3}$ where $C(t)$ is concentration in mg/L and t is time in hours.

- a) What is the initial concentration?
- b) Find the half-life (time for concentration to reduce by half).
- c) After how many hours will the concentration be 2 mg/L?
- d) Express the time t as a function of concentration C .

i Solution to Task 3

Part a: Initial concentration

$$C(0) = 64 \cdot 2^0 = 64 \text{ mg/L}$$

Part b: Half-life

Step 1: Set up equation for half of initial concentration

$$32 = 64 \cdot 2^{-t/3}$$

Step 2: Simplify

$$\frac{1}{2} = 2^{-t/3}$$

$$2^{-1} = 2^{-t/3}$$

Step 3: Equate exponents

$$-1 = -\frac{t}{3}$$

$$t = 3$$

Answer: Half-life is 3 hours

Part c: When concentration is 2 mg/L

Step 1: Set up equation

$$2 = 64 \cdot 2^{-t/3}$$

Step 2: Simplify

$$\frac{2}{64} = 2^{-t/3}$$

$$\frac{1}{32} = 2^{-t/3}$$

$$2^{-5} = 2^{-t/3}$$

Step 3: Equate exponents

$$-5 = -\frac{t}{3}$$

$$t = 15$$

Answer: After 15 hours

Part d: Express t as function of C

Step 1: Start with $C = 64 \cdot 2^{-t/3}$

Step 2: Isolate the exponential term

$$\frac{C}{64} = 2^{-t/3}$$

Step 3: Take logarithm base 2

$$4$$

$$\log_2 \left(\frac{C}{64} \right) = -\frac{t}{3}$$

Task 4: Rational Function Analysis [xxx]

Analyze the function $f(x) = \frac{x^2-9}{x^2-4x+3}$.

- Factor completely and identify any holes.
- Find all vertical asymptotes.
- Determine the horizontal asymptote.
- Find all intercepts.
- Sketch the function, clearly marking all features.

i Solution to Task 4

Part a: Factor and find holes

Step 1: Factor numerator and denominator - Numerator: $x^2 - 9 = (x - 3)(x + 3)$ - Denominator: $x^2 - 4x + 3 = (x - 3)(x - 1)$

Step 2: Identify common factors

$$f(x) = \frac{(x-3)(x+3)}{(x-3)(x-1)}$$

Common factor: $(x - 3)$ creates a hole at $x = 3$

Step 3: Find hole coordinates Simplified form: $f(x) = \frac{x+3}{x-1}$ for $x \neq 3$ At $x = 3$: $y = \frac{3+3}{3-1} = \frac{6}{2} = 3$ Hole at $(3, 3)$

Part b: Vertical asymptotes

From simplified form: $x - 1 = 0$ gives $x = 1$ Vertical asymptote at $x = 1$

Part c: Horizontal asymptote

Original function has equal degrees (both 2) Leading coefficients: numerator = 1, denominator = 1 Horizontal asymptote: $y = 1$

Part d: Intercepts

x-intercepts: Set numerator = 0 in simplified form $x + 3 = 0$ gives $x = -3$

y-intercept: $f(0) = \frac{0+3}{0-1} = -3$

Intercepts: $(-3, 0)$ and $(0, -3)$

Part e: Key features for sketch

- Vertical asymptote: $x = 1$ (dashed line)
- Horizontal asymptote: $y = 1$ (dashed line)
- Hole: $(3, 3)$ (open circle)
- Intercepts: $(-3, 0)$ and $(0, -3)$

Task 5: Trigonometric Modeling [xxx]

The height of a Ferris wheel car above ground is modeled by:

$$h(t) = 25 - 20 \cos\left(\frac{\pi t}{30}\right)$$

where h is in meters and t is time in seconds.

- Find the amplitude, period, and vertical shift.
- What are the maximum and minimum heights?
- At what times in the first minute is the car at exactly 25 meters?
- How long does it take to go from minimum to maximum height?

i Solution to Task 5

Part a: Amplitude, period, vertical shift

- **Amplitude:** $|-20| = 20$ meters
- **Period:** $\frac{2\pi}{\pi/30} = \frac{2\pi \cdot 30}{\pi} = 60$ seconds
- **Vertical shift:** 25 meters (upward)

Part b: Maximum and minimum heights

- **Maximum:** When $\cos = -1$: $h = 25 - 20(-1) = 45$ meters
- **Minimum:** When $\cos = 1$: $h = 25 - 20(1) = 5$ meters

Part c: When height is 25 meters

Step 1: Set up equation

$$25 - 20 \cos\left(\frac{\pi t}{30}\right) = 25$$

$$-20 \cos\left(\frac{\pi t}{30}\right) = 0$$

$$\cos\left(\frac{\pi t}{30}\right) = 0$$

Step 2: Solve for t

$$\frac{\pi t}{30} = \frac{\pi}{2} + n\pi$$

$$t = 15 + 30n$$

Step 3: Find times in first minute - $n = 0$: $t = 15$ seconds - $n = 1$: $t = 45$ seconds

Answer: At 15 seconds and 45 seconds

Part d: Time from minimum to maximum

The minimum occurs when $\cos = 1$, maximum when $\cos = -1$. This is half a period:
 $\frac{60}{2} = 30$ seconds

Task 6: Logarithmic Applications [xxx]

The pH of a solution is given by $pH = -\log_{10}[H^+]$ where $[H^+]$ is hydrogen ion concentration in moles per liter.

- a) If a solution has $pH = 3.5$, find the hydrogen ion concentration.
- b) If the concentration doubles, by how much does the pH change?
- c) What concentration gives a neutral pH of 7?
- d) Express $[H^+]$ as a function of pH.

i Solution to Task 6

Part a: Find concentration at pH = 3.5

Step 1: Use the pH formula

$$3.5 = -\log_{10}[H^+]$$

$$-3.5 = \log_{10}[H^+]$$

Step 2: Convert to exponential form

$$[H^+] = 10^{-3.5} = \frac{1}{10^{3.5}} = \frac{1}{\sqrt{10^7}} = \frac{1}{\sqrt{10,000,000}}$$

$$[H^+] = 10^{-3.5} \approx 3.16 \times 10^{-4}$$

moles/L

Part b: pH change when concentration doubles

Step 1: Original pH

$$pH_1 = -\log_{10}[H^+]$$

Step 2: New pH with doubled concentration

$$pH_2 = -\log_{10}(2[H^+]) = -\log_{10}(2) - \log_{10}[H^+]$$

$$pH_2 = -\log_{10}(2) + pH_1$$

Step 3: Change in pH

$$\Delta pH = pH_2 - pH_1 = -\log_{10}(2) \approx -0.301$$

Answer: pH decreases by approximately 0.301 units

Part c: Concentration for neutral pH

$$7 = -\log_{10}[H^+]$$

$$[H^+] = 10^{-7}$$

moles/L

Part d: Express concentration as function of pH

Step 1: Start with $pH = -\log_{10}[H^+]$

Step 2: Solve for $[H^+]$

$$-pH = \log_{10}[H^+]$$

$$[H^+] = 10^{-pH}$$

Task 7: Function Comparison [xxxx]

Consider three investment strategies with values after t years: - Strategy A: $A(t) = 10000(1.05)^t$ (5% annual compound interest) - Strategy B: $B(t) = 10000 + 600t$ (linear growth) - Strategy C: $C(t) = 10000\sqrt{1 + 0.1t}$ (square root growth)

- a) Which strategy has the highest value after 10 years?
- b) Find when Strategy A overtakes Strategy B.
- c) Prove that Strategy A eventually exceeds both other strategies for large t .
- d) Find the average rate of change for each strategy over the first 5 years.

i Solution to Task 7

Part a: Values after 10 years

- $A(10) = 10000(1.05)^{10} = 10000(1.6289) = 16,289$
- $B(10) = 10000 + 600(10) = 16,000$
- $C(10) = 10000\sqrt{1+1} = 10000\sqrt{2} \approx 14,142$

Answer: Strategy A has the highest value at \$16,289

Part b: When A overtakes B

Step 1: Set up equation

$$10000(1.05)^t = 10000 + 600t$$

$$(1.05)^t = 1 + 0.06t$$

Step 2: Test values - At $t = 15$: $(1.05)^{15} = 2.079$ vs $1 + 0.9 = 1.9 \rightarrow$ A wins - At $t = 14$: $(1.05)^{14} = 1.980$ vs $1 + 0.84 = 1.84 \rightarrow$ A wins - At $t = 13$: $(1.05)^{13} = 1.886$ vs $1 + 0.78 = 1.78 \rightarrow$ A wins - At $t = 12$: $(1.05)^{12} = 1.796$ vs $1 + 0.72 = 1.72 \rightarrow$ A wins - At $t = 11$: $(1.05)^{11} = 1.710$ vs $1 + 0.66 = 1.66 \rightarrow$ A wins - At $t = 10$: $(1.05)^{10} = 1.629$ vs $1 + 0.6 = 1.6 \rightarrow$ A wins

More precisely, A overtakes B between year 9 and 10.

Part c: Long-term dominance of A

Proof: - $\lim_{t \rightarrow \infty} \frac{A(t)}{B(t)} = \lim_{t \rightarrow \infty} \frac{10000(1.05)^t}{10000+600t} = \infty$ (exponential beats linear) - $\lim_{t \rightarrow \infty} \frac{A(t)}{C(t)} = \lim_{t \rightarrow \infty} \frac{10000(1.05)^t}{10000\sqrt{1+0.1t}} = \infty$ (exponential beats square root)

Therefore, Strategy A eventually dominates both others.

Part d: Average rates of change over first 5 years

Formula: Average rate = $\frac{f(5)-f(0)}{5-0}$

Strategy A:

$$\frac{A(5) - A(0)}{5} = \frac{10000(1.05)^5 - 10000}{5} = \frac{10000(1.2763 - 1)}{5} = 552.60$$

Strategy B:

$$\frac{B(5) - B(0)}{5} = \frac{13000 - 10000}{5} = 600$$

Strategy C:

$$\frac{C(5) - C(0)}{5} = \frac{10000\sqrt{1.5} - 10000}{5} = \frac{10000(1.2247 - 1)}{5} = 449.40$$

Answer: Average rates: A = \$552.60/year, B = \$600/year, C = \$449.40/year

Exam Readiness Checklist

After completing these tasks, you should be able to:

- ☐ Model exponential growth and decay
- ☐ Apply function transformations systematically
- ☐ Solve logarithmic equations
- ☐ Analyze rational functions completely
- ☐ Work with trigonometric models
- ☐ Solve optimization problems
- ☐ Compare different function types