Understanding the Quadratic Formula & Discriminant

A detailed Explanation

Introduction

The quadratic formula is one of the most powerful tools in algebra. It can solve ANY quadratic equation, guaranteed! Let's discover where it comes from and how to use it effectively.

Part 1: Deriving the Quadratic Formula

Starting Point

We want to solve any equation of the form:

$$ax^2 + bx + c = 0$$

where $a \neq 0$ (otherwise it wouldn't be quadratic!)

The Derivation (Step by Step)

Step 1: Divide by a

Since $a \neq 0$, we can safely divide everything by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2: Move the constant term

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step 3: Complete the Square

To complete the square on the left side, we need to add $\left(\frac{b}{2a}\right)^2$ to both sides.

> **i Why this number?**
>
> Because $(x + \square)^2 = x^2 + 2\square x + \square^2$

We have $\frac{b}{a}x$, so $2\square = \frac{b}{a}$, which means $\square = \frac{b}{2a}$

Adding to both sides:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Step 4: Factor the left side

The left side is now a perfect square:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Step 5: Simplify the right side

To combine the fractions on the right, we need a common denominator:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

Step 6: Take the square root

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Step 7: Solve for x

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is the Quadratic Formula!

Part 2: The Discriminant

What is the Discriminant?

The discriminant is the part under the square root:

$$\Delta = b^2 - 4ac$$

It "discriminates" between different types of solutions.

Why Does It Matter?

Before solving an equation, the discriminant tells you:

- How many real solutions exist
- What type of numbers the solutions will be
- Whether the quadratic factors nicely

Part 3: Cases of the Discriminant

Case 1: $\Delta > 0$ (Positive Discriminant)

What happens mathematically:

- The square root of a positive number gives a real result
- The $\pm$ gives two different values

- You get two distinct real solutions

Example:
Solve $x^2 - 5x + 6 = 0$

- $a = 1, b = -5, c = 6$
- $\Delta = (-5)^2 - 4(1)(6) = 25 - 24 = 1$
- Since $\Delta = 1 > 0$, we expect two real solutions
- $x = \frac{5 \pm \sqrt{1}}{2} = \frac{5 \pm 1}{2}$
- Solutions: $x = 3$ or $x = 2$

Special Subcase: Δ is a perfect square
When Δ is a perfect square $(1, 4, 9, 16, 25, \ldots)$, the solutions are rational and the quadratic factors nicely.

In our example: $x^2 - 5x + 6 = (x - 2)(x - 3)$

Hence, the parabola crosses the x-axis at two points.

Case 2: $\Delta = 0$ (Zero Discriminant)

What happens mathematically:
- $\sqrt{0} = 0$, so the $\pm$ doesn't create two different values
- You get one repeated real solution (or a "double root")

Example:
Solve $x^2 - 6x + 9 = 0$

- $a = 1, b = -6, c = 9$
- $\Delta = (-6)^2 - 4(1)(9) = 36 - 36 = 0$
- $x = \frac{6 \pm \sqrt{0}}{2} = \frac{6}{2} = 3$
- One solution: $x = 3$ (repeated)

Factored Form:
When $\Delta = 0$, the quadratic is a perfect square: $x^2 - 6x + 9 = (x - 3)^2$

Hence, the parabola just touches the x-axis at one point (the vertex).

Case 3: $\Delta < 0$ (Negative Discriminant)

What happens mathematically:
- You can't take the square root of a negative number (in real numbers)
- You get no real solutions
- The solutions are complex conjugates of the form $a + bi$ and $a - bi$

Example:
Solve $x^2 - 2x + 5 = 0$

- $a = 1, b = -2, c = 5$
- $\Delta = (-2)^2 - 4(1)(5) = 4 - 20 = -16$
- Since $\Delta < 0$, no real solutions exist

Hence, the parabola doesn't touch the x-axis at all.

Decision Flowchart

```
Quadratic Equation: ax² + bx + c = 0
│
├─ Calculate Δ = b² - 4ac
│
├─ Δ < 0 → No real solutions
│
├─ Δ = 0 → One solution: x = -b/(2a) (Perfect square trinomial)
│
└─ Δ > 0 → Two real solutions
           │
           └─ Is Δ a perfect square?
              │
              ├─ YES → Try factoring first
              │
              └─ NO → Use quadratic formula
```

Key Takeaways

1. The quadratic formula always works - it's your reliable backup plan
2. The discriminant predicts your solutions before you solve
3. Perfect square discriminants mean nice factorization
4. Zero discriminant means a perfect square trinomial
5. Negative discriminant means complex solutions (no x-intercepts)