Compound Growth & Interest

What is Compounding?

Compounding means earning returns on your returns. Each period, you earn interest not just on your original amount, but also on all previously earned interest.

Simple Example: If you invest $100 at 10% annual interest:

- Year 1: $100 → $110 (earned $10)
- Year 2: $110 → $121 (earned $11, not just $10!)
- Year 3: $121 → $133.10 (earned $12.10)

Part 1: The Three Main Compounding Formulas

1. Annual/Period Compounding (Most Common)

Formula: $A = P(1 + r)^t$

Where:

- A = Final amount
- P = Principal (starting amount)
- r = Rate per period (as decimal: 8% = 0.08)
- t = Number of periods

When to use: When interest compounds once per period (annually, monthly, etc.)

Example: $1,000 invested at 6% annual interest for 5 years

- $A = 1000(1.06)^5

2. Multiple Compounding Per Year

Formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Where:

- r = Annual nominal rate (as decimal)
- n = Compounding frequency per year
  - Monthly: $n = 12$
  - Quarterly: $n = 4$
  - Daily: $n = 365$
- t = Time in years

When to use: When interest compounds multiple times per year

Example: $5,000 at 8% annual rate, compounded monthly for 3 years

- $A = 5000(1 +)^{12 }

3. Continuous Compounding

Formula: $A = Pe^{rt}$

Where:

- $e \approx$ 2.71828 (Euler's number)
- Other variables are basically the same as above

When to use: Theoretical maximum compounding

Example: $2,000 at 7% continuously compounded for 10 years

- $A = 2000e^{0.07 }

Part 2: Comparing Methods

Effective Annual Rate (EAR)

The EAR shows the actual yearly return when compounding occurs more frequently.

From periodic compounding: $\text{EAR} = \left(1 + \frac{r}{n}\right)^n - 1$

From continuous: $\text{EAR} = e^r - 1$

Example: 12% nominal rate

Compounding	EAR Calculation	Result
Annual	12%	12.00%
Quarterly	$(1.03)^4 - 1$	12.55%
Monthly	$(1.01)^{12} - 1$	12.68%
Continuous	$e^{0.12} - 1$	12.75%

Part 3: General Exponential Growth

Population/Bacteria Growth

If something multiplies by factor k every Δt time units:

Formula: $P(t) = P_0 \cdot k^{t/\Delta t}$

Example: Bacteria triple every 4 hours, starting with 500

- After 12 hours: $P(12) = 500 \cdot 3^{12/4} = 500 \cdot 27 = 13,500$ bacteria

Converting to continuous rate: $r = \frac{\ln(k)}{\Delta t}$

- For tripling every 4 hours: $r = \frac{\ln(3)}{4} = 0.275$ per hour

> **i Note**
>
> That's the important part that made the task in lecture 01-06 difficult! We know, that the bacteria triples every 4 hours. That way, we can essentially either work with continous compounding or with periodic compounding, as long as the bacteria triples every 4 hours.
>
> - Tripling every 4 hours: $A = P \cdot 3^{t/4}$
> - Equivalent continuous form: $A = P \cdot e^{\ln(3)/4)t}$
> - They match because $3^{t/4} = e^{(\ln 3/4)t}$

Practice Problems

Basic Application

1. You invest $3,000 at 5% annual interest. What's the value after 8 years?
2. A savings account offers 4.8% compounded monthly. If you deposit $2,500, what's the value after 3 years?
3. How long does it take $1,000 to grow to $2,500 at 9% annual interest?
4. A card charges 18% APR compounded daily. If you owe $5,000 and make no payments, what will you owe after 2 years?
5. A city of 50,000 grows at 3% annually. What's the population after 20 years?
6. You need $50,000 in 10 years for a down payment. If you can earn 7% annually, how much must you invest today?
7. A bacteria culture doubles every 3 hours. If you start with 1,000 bacteria at noon, how many will there be at midnight?