Session 09-02: Tasks

Differential Calculus - Exam Review

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Problem 1: Limit Evaluation (x)

Evaluate the following limits:

a) $\lim_{x \to 0} \frac{e^{5x}-1}{x}$

b) $\lim_{n \to \infty} \left(1 + \frac{3}{n}\right)^n$

c) $\lim_{x \to \infty} \frac{4x^2-x+3}{2x^2+5}$

d) $\lim_{x \to 2} \frac{x^2-4}{x-2}$

Problem 2: Differentiation Drill (x)

Find $f'(x)$ for each function:

a) $f(x) = 3x^4 - 2x^3 + 7x - 1$

b) $f(x) = e^{3x+1}$

c) $f(x) = \ln(2x+5)$

d) $f(x) = \frac{x^2+1}{x}$

e) $f(x) = (3x-1)^5$

f) $f(x) = x \cdot e^x$

Problem 3: Basic Critical Points (x)

Find all critical points and classify them using the second derivative test:

a) $f(x) = x^3 - 12x + 5$

b) $f(x) = -2x^2 + 8x - 3$

Problem 4: Critical Point Classification (xx)

For $f(x) = x^4 - 8x^2 + 16$:

a) Find all critical points.

b) Use the second derivative test to classify them. Handle the case where $f''(x_0) = 0$ carefully.

c) Find the global extrema on the interval $[-3, 3]$.

d) Determine the intervals of increase and decrease.

Problem 5: Quadratic Optimization Word Problem (xx)

A rectangular garden is to be fenced on three sides (the fourth side is along a wall). The total available fencing is 120 meters.

a) Express the area as a function of one variable.

b) Find the dimensions that maximize the area.

c) What is the maximum area?

d) Verify using the second derivative test.

Problem 6: Polynomial Curve Sketch (xx)

For $f(x) = x^3 - 3x^2 - 9x + 27$:

a) Find the zeros (factor the polynomial).

b) Find the critical points and classify them.

c) Find the inflection points.

d) Determine the end behavior.

e) Sketch the complete graph.

Problem 7: Tangent Line at a Point (xx)

Consider the function $f(x) = x^2 \cdot e^{-x}$.

a) Find the equation of the tangent line to f at $x = 2$.

b) Find the equation of the normal line at the same point.

c) At which point(s) on the curve is the tangent line horizontal?

Problem 8: Rational Curve Sketch with Asymptotes (xxx)

For $f(x) = \frac{x^2-2x-3}{x-2}$:

a) Determine the domain.

b) Find the vertical asymptote(s).

c) Perform polynomial long division to find the oblique asymptote.

d) Find all zeros and the y-intercept.

e) Compute $f'(x)$ and find the critical points.

f) Compute $f''(x)$ and find the inflection points.

g) Sketch the complete graph.

Problem 9: Funktionsscharen Parameter Analysis (xxx)

For $f_t(x) = x^3 - 3t^2x + 2t^3$ with $t > 0$:

a) Show that $x = t$ is always a zero of f_t.

b) Factor $f_t(x)$ completely.

c) Find all extrema as a function of t.

d) For which value of t does the local minimum have a y-value of -16?

Problem 10: Curve Sketch of $f(x) = x \cdot \ln(x)$ (xxx)

a) Determine the domain.

b) Find the zeros.

c) Compute $\lim_{x \to 0^+} x \ln(x)$. (Hint: rewrite as $\frac{\ln(x)}{1/x}$ and apply L'Hôpital's rule.)

d) Find $f'(x)$ and the critical points.

e) Find $f''(x)$ and the inflection points.

f) Sketch the complete graph.

g) Find the minimum value and interpret it.

Problem 11: Tangent Line System (xxx)

The tangent line to $f(x) = ax^2 + bx + 1$ at $x = 1$ has the equation $y = 3x - 1$.

a) Set up a system of equations using $f(1)$ and $f'(1)$.

b) Solve for a and b.

c) Sketch $f(x)$ and the tangent line.

Problem 12: Cost Minimization with e-Function (xxx)

A company's production cost per unit is modeled by:

$$C(x) = 50 + \frac{200}{x} + 0.1x \cdot e^{0.01x}$$

where $x > 0$ is the number of units produced daily.

a) What is the cost per unit when $x = 50$ units are produced?

b) Find $C'(x)$.

c) Show that there exists a production level that minimizes the cost per unit. (Argue using the behavior as $x \to 0^+$ and $x \to \infty$.)

d) Use a numerical/graphical argument to estimate the optimal production level.

Problem 13: Oblique Asymptote + Full Sketch (xxx)

For $f(x) = \frac{2x^2 - x - 6}{x + 2}$:

a) Perform polynomial long division.

b) Identify the oblique and vertical asymptotes.

c) Find all zeros.

d) Find $f'(x)$ and the critical points.

e) Sketch the graph with the oblique asymptote.

Problem 14: Related Rates Word Problem (xxx)

A spherical balloon is inflated so that its volume increases at a constant rate of $100\ \mathrm{cm}^3/\mathrm{s}$.

a) Express the volume as a function of the radius: $V = \frac{4}{3}\pi r^3$.

b) Find $\frac{dr}{dt}$ when $r = 10$ cm.

c) Find $\frac{dA}{dt}$ when $r = 10$ cm, where $A = 4\pi r^2$ is the surface area.

d) At what radius is the surface area increasing at a rate of $40\ \mathrm{cm}^2/\mathrm{s}$?

Problem 15: Advertising Effectiveness Full Analysis (xxxx)

An advertising campaign's effectiveness is modeled by:

$$E(t) = 500(1 - e^{-0.3t})$$

where $E(t)$ is in thousands of impressions reached and t is weeks since the campaign started.

a) Find $E(0)$ and interpret the result.

b) Find $\lim_{t\to\infty} E(t)$ and interpret.

c) Find $E'(t)$ and interpret $E'(0)$.

d) Find $E''(t)$. Is the campaign's effectiveness concave up or concave down? Interpret this in the context of advertising.

e) The campaign costs €2,000 per week. Revenue per thousand impressions is €8. Find the time t^* at which marginal revenue equals marginal cost.

f) Calculate the total profit at t^*.

Problem 16: Tangent from External Point (xxxx)

Find all tangent lines from the point $P(0, -4)$ to the curve $f(x) = x^3 - 3x$.

a) Let the tangent touch the curve at $(a, f(a))$. Write the tangent line equation.

b) Use the condition that the tangent passes through $P(0, -4)$ to obtain an equation in a.

c) Solve for a.

d) Write the equations of all tangent lines.

e) Sketch the curve and all tangent lines.

Problem 17: Funktionsscharen with Area Constraint (xxxx)

For $f_a(x) = ax^2 - 2ax$ with $a > 0$:

a) Find the zeros and the vertex as functions of a.

b) Sketch $f_1(x)$ and $f_2(x)$.

c) The region between $f_a(x)$ and the x-axis has area $\frac{4a}{3}$. Verify this by computing the definite integral.

d) Find a such that the enclosed area equals 12.

e) Find the tangent line to f_a at $x = 0$ as a function of a. For which value of a does this tangent line pass through the point $(3, -12)$?