Practice Tasks - Session 05-06

Optimization & Curve Sketching

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Part 1: First and Second Derivative Tests

Problem 1: First Derivative Test (xx)

Find all critical points of $f(x) = x^3 - 3x^2 - 9x + 5$ and classify each using the first derivative test.

Problem 2: Second Derivative Test (xx)

Use the second derivative test to classify all critical points of:

$$g(x) = 2x^3 - 9x^2 + 12x - 3$$

Problem 3: When Second Derivative Test Fails (xxx)

For $h(x) = x^4 - 4x^3$:

a) Find all critical points.

b) Attempt to classify them using the second derivative test.

c) For any critical point where the second derivative test fails, use the first derivative test.

Problem 4: Comparing Both Tests (xx)

For $f(x) = x^4 - 8x^2 + 5$:

a) Find all critical points.

b) Classify them using the second derivative test.

c) Verify your classifications using the first derivative test.

Part 2: Global Extrema on Intervals

Problem 5: Finding Absolute Extrema (xx)

Find the absolute maximum and minimum values of $f(x) = x^3 - 6x^2 + 9x + 1$ on the interval $[0, 4]$.

Problem 6: Closed Interval Method (xx)

Find the absolute extrema of $g(x) = \frac{x}{x^2+1}$ on $[-2, 2]$.

Problem 7: Optimization with Constraints (xxx)

A rectangular box with a square base and no top is to have a volume of 32 cubic meters. Find the dimensions that minimize the surface area.

Problem 8: Absolute Extrema with Domain Restrictions (xxx)

Find the absolute extrema of $f(x) = xe^{-x}$ on $[0, 3]$.

(Note: You may use the fact that $(e^{-x})' = -e^{-x}$)

Part 3: Complete Curve Sketching

Problem 9: Sketching a Rational Function (xxx)

Use the 6-step algorithm to sketch $f(x) = \frac{x^2-4}{x}$.

a) Domain and intercepts

b) Critical points

c) Inflection points

d) Sign charts for f' and f''

e) Asymptotes

f) Complete sketch

Problem 10: Sketching a Polynomial (xxx)

Use the 6-step algorithm to sketch $g(x) = x^4 - 4x^3 + 4x^2$.

Problem 11: Challenging Rational Function (xxxx)

Sketch $h(x) = \frac{x^2}{x^2-4}$ using the complete algorithm.

Part 4: Business Optimization

Problem 12: Profit Maximization (xx)

A company's profit function (in thousands of euros) is:

$$P(x) = -x^3 + 12x^2 - 36x + 50$$

where x is production level (in thousands of units).

a) Find the production level that maximizes profit.

b) What is the maximum profit?

c) For what production levels is the company making a profit (i.e., $P(x) > 0$)?

Problem 13: Cost Minimization (xxx)

The average cost per unit for a manufacturer is:

$$\bar{C}(x) = 0.01x^2 - 0.6x + 13 + \frac{50}{x}$$

where x is the number of units produced (in hundreds).

a) Find the production level that minimizes average cost.

b) What is the minimum average cost?

c) Verify that your answer is indeed a minimum.

Problem 14: Revenue Maximization (xx)

A company can sell x thousand units at a price of $p = 100 - 2x$ euros per unit.

a) Write the revenue function $R(x)$.

b) Find the production level that maximizes revenue.

c) What is the maximum revenue?

d) What price should be charged to achieve maximum revenue?

Problem 15: Optimization with Constraint (xxxx)

A farmer has 100 meters of fencing and wants to enclose a rectangular area next to a river (so fencing is needed on only three sides).

a) Express the area A as a function of the width x perpendicular to the river.

b) What dimensions maximize the enclosed area?

c) What is the maximum area?

Problem 16: Inventory Optimization (xxxx)

A store sells 1000 units per year of a certain product. The ordering cost is €20 per order, and the holding cost is €5 per unit per year.

The Economic Order Quantity (EOQ) model gives the total annual cost as:

$$C(x) = \frac{1000 \cdot 20}{x} + \frac{x \cdot 5}{2} = \frac{20000}{x} + 2.5x$$

where x is the order size.

a) Find the order size that minimizes total cost.

b) What is the minimum total annual cost?

c) How many orders should be placed per year?

Important:

1. Always verify critical points using derivative tests

2. Don't forget to check endpoints when finding global extrema
3. The 6-step algorithm provides a systematic approach to curve sketching
4. Business problems require careful setup of objective functions and constraints
5. Both first and second derivatives provide essential information about function behavior

Exam Preparation:

- Practice the curve sketching algorithm until it becomes automatic
- Master both derivative tests and know when to use each
- Always verify that your critical points are actually maxima or minima!