Tasks 05-02 - The Derivative as Rate of Change

Section 05: Differential Calculus

Problem 1: Computing Derivatives from Definition (x)

Use the limit definition $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ to find the derivative at the given point:

a) $f(x) = 5x - 3$ at $a = 2$

b) $f(x) = x^2 + 2x$ at $a = 1$

c) $f(x) = 4$ at $a = 3$

d) Explain what the derivative value tells you about each function's behavior at the given point.

Problem 2: Average vs. Instantaneous Rate of Change (x)

A ball is thrown upward. Its height (in meters) after t seconds is given by:

$$h(t) = -5t^2 + 20t + 2$$

a) Compute the average rate of change of height from $t = 1$ to $t = 3$ seconds.

b) Estimate the instantaneous rate of change at $t = 2$ by calculating $\frac{h(2.01) - h(2)}{0.01}$.

c) Verify your answer from (b) by computing $h'(2)$ using the limit definition.

d) Interpret the meaning of $h'(2)$ in the context of the ball's motion.

Problem 3: Marginal Cost Analysis (xx)

A manufacturer's total cost function (in euros) for producing x units is:

$$C(x) = 500 + 10x + 0.05x^2$$

a) Determine the average cost per unit when producing 100 units.

b) Compute $C'(100)$ by calculating $C(101) - C(100)$.

c) Investigate the behavior of marginal cost: Is it increasing, decreasing, or constant? Calculate $C'(50)$ and $C'(150)$ using the approximation method.

d) Argue whether the company should expand from 100 to 101 units if they can sell each unit for €25.

Problem 4: Revenue and Marginal Revenue (xx)

A company sells widgets. The demand function is $p = 80 - 0.4x$, where p is the price per widget (in euros) and x is the quantity demanded.

a) Show that the revenue function is $R(x) = 80x - 0.4x^2$.

b) Compute $R'(75)$ using the approximation $R(76) - R(75)$.

c) Assess whether the company should increase or decrease production if they are currently producing 75 widgets. Substantiate your answer.

d) Decide: At what production level is revenue maximized? (Hint: Find where $R'(x) = 0$ by using the fact that $R'(x) = 80 - 0.8x$.)

Problem 5: Profit Optimization (xx)

A small business has the following functions: - Cost: $C(x) = 200 + 8x + 0.02x^2$ - Revenue: $R(x) = 50x - 0.3x^2$

where x is the number of items produced and sold.

a) Determine the profit function $P(x) = R(x) - C(x)$.

b) Compute the marginal profit at $x = 40$ using $P(41) - P(40)$.

c) Give the production level that maximizes profit using the formula $P'(x) = R'(x) - C'(x)$ and the fact that:

- $R'(x) = 50 - 0.6x$
- $C'(x) = 8 + 0.04x$

d) Verify that this production level gives $MR = MC$.

e) Graph the profit function and mark the optimal production point.

Problem 6: Advanced Derivative Computation (xxx)

a) Use the limit definition to find $f'(x)$ as a function of x (not just at a specific point) for $f(x) = x^2 + 3x$.

b) Show that your answer from (a) satisfies the property that $f'(2) = 7$ by direct evaluation.

c) Investigate: For what value of x is the tangent line to f horizontal? Give the point on the curve where this occurs.

d) Verify by computing $f'(1)$ and finding the equation of the tangent line at $(1, f(1))$.

Problem 7: Comprehensive Business Optimization (xxx)

A tech startup develops mobile apps. Their analysis shows:

- Development cost: $C(x) = 5000 + 800x + 20x^2$ (in euros)
- Revenue model: $R(x) = 2000x - 15x^2$ (in euros)

where x is the number of apps developed per month.

Part A: Function Analysis

a) Determine the profit function $P(x)$.

b) Compute the average cost per app when developing 20 apps per month.

c) Show that the marginal cost and marginal revenue at $x = 20$ are:

- $MC(20) \approx 1200$ euros/app
- $MR(20) \approx 1400$ euros/app

Part B: Optimization

d) Decide whether the company should increase or decrease production from 20 apps. Substantiate mathematically.

e) Investigate the optimal production level by finding where $P'(x) = 0$, using:

- $R'(x) = 2000 - 30x$
- $C'(x) = 800 + 40x$

f) Give the maximum profit and verify that $MR = MC$ at this production level.

Part C: Strategic Analysis

g) Argue whether it's worth developing apps if the startup can only manage 10 apps per month.

h) Assess the break-even points (where $P(x) = 0$) and interpret their business significance.