Tasks 01-06 - Synthesis & Problem Solving

Integrating Mathematical Foundations

Problem 1: Multi-Technique Integration

Solve the following problems using multiple techniques:

a) Solve for x: $\log_2(x^2 - 9) = \log_2(x - 3) + 3$ (very hard and too early, sorry!)

b) Factor completely, then evaluate at $x = \sqrt{2}$: $x^4 - 5x^2 + 4$

c) Simplify the expression $\frac{(x^3-8)\cdot\sqrt{x+2}}{(x-2)\cdot(x^2+2x+4)}$ when $x = 6$

d) If $2^x = 3$ and $3^y = 4$, find the value of 6^{xy}

e) Expand $\left(1 + \sqrt{2}\right)^3$ and express in the form $a + b\sqrt{2}$

Problem 2: Exponential-Logarithmic Systems

Solve the following systems and equations:

a) System: $2^x \cdot 3^y = 72$ and $x = 3$ (find y, then verify with logarithms)

b) If $\log_a(b) = 2$ and $\log_b(c) = 3$, find $\log_a(c)$ and $\log_c(a)$

c) A bacteria culture doubles every 3 hours. Starting with 1000 bacteria, the number after t hours = $1000 \cdot 2^{t/3}$. After how many hours will there be exactly 10^6 bacteria?

Problem 3: Complex Integration Problem

A pharmaceutical company is developing a new drug with the following characteristics:

Concentration: After t hours, concentration = $100 \cdot 2^{-t/4} + 20$ mg/L

Production Cost: Cost to produce x units = $1000 + 50x + 0.1x^2$ euros

Market Demand: At price p euros, demand = $10000 \cdot 0.95^p$ units

a) Find when the concentration drops to half its initial value.

b) At what time does the concentration reach exactly 30 mg/L?

c) What is the production cost for 100 units? Express in scientific notation.

d) If the company wants demand of exactly 5000 units, what price should they set?

Problem 4: Advanced Synthesis

Solve these challenging integration problems:

a) If $y = \sqrt{x + y}$ and $y = 3$, find x.

b) Simplify: $\frac{\log_2(32) - \log_4(64) + \log_8(512)}{\log_{16}(256)}$

c) If $a^3 + b^3 = 9$ and $a + b = 3$, find ab.

Problem 5: Challenge - Research Project

A research institute plans a 5-year project:

Funding: Initial €1 million, growing each year by factor 1.08

- Amount after t years = $1000000 \cdot 1.08^t$

Papers Published: Cumulative papers after t years = $5 \cdot \ln(t+1) + 2t$ (Use $\ln(2) \approx 0.69$, $\ln(3) \approx 1.10$, $\ln(4) \approx 1.39$)

Annual Cost: Cost in year t = $200000 \cdot 1.05^t + 50000\sqrt{t+1}$

a) What is the funding amount after 5 years? Express in scientific notation.

b) How many papers will be published by the end of year 3?

c) Calculate the total cost for years 0, 1, and 2.