Session 09-07 - Final Recap: Function Analysis & Probability

Section 09: Exam Preparation

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Concept Refresher - Function Analysis - 25 Minutes

Verifying Extrema (Second Derivative)

To prove that f has a maximum or minimum at $x = a$:

1. Show $f'(a) = 0$ (stationary point)
2. Compute $f''(a)$:
   - $f''(a) < 0 \Rightarrow$ **local maximum**
   - $f''(a) > 0 \Rightarrow$ **local minimum**
   - $f''(a) = 0 \Rightarrow$ inconclusive; check sign of f'
3. Compute $f(a)$ to give the actual extremum value

. . .

> Tip
>
> The exam often asks: *"Verify computationally that f has a maximum at $x = a$."* All three steps are needed for full points.

Symmetry & Reflections

Transformation	Formula	Meaning
Reflection at y-axis	$f(-x)$	Mirror left/right
Reflection at x-axis	$-f(x)$	Mirror up/down
Vertical shift	$f(x) + c$	Move up by c
Horizontal shift	$f(x - c)$	Move right by c

. . .

Quick: If $f(x) = 2x(x - 3)^2 - 4$, then the reflection at the y-axis is?

. . .

$$f_1(x) = f(-x) = 2(-x)(-x - 3)^2 - 4 = -2x(x + 3)^2 - 4$$

Areas Between Curves

Steps:

1. Find **intersection points**: solve $f(x) = g(x)$
2. Form the **difference** $h(x) = f(x) - g(x)$
3. Determine which function is **on top** in each interval (test a value)
4. Integrate $|h(x)|$ over each region; areas are always **positive**

...

$$A = \int_a^b |\, f(x) - g(x)|\, dx$$

...

> ! Important
>
> If h changes sign, **split** the integral at each zero. Otherwise the positive and negative parts cancel.

Areas Between Curves: Example

Find the area enclosed between $f(x) = x^2$ and $g(x) = 2x$ on $[0, 2]$:

...

Step 1, intersections: $x^2 = 2x \Rightarrow x(x-2) = 0 \Rightarrow x = 0, 2$

...

Step 2, top function: at $x = 1$: $g(1) = 2 > 1 = f(1)$; so g is on top.

...

$$A = \int_0^2 (2x - x^2)\, dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

True/False Statements From Graphs

To assess claims about f, f', f'' from a graph, recall:

- $f'(a) > 0 \Leftrightarrow f$ is **increasing** at a
- $f''(a) > 0 \Leftrightarrow f$ is **concave up** at a (smiley)
- $f''(a) < 0 \Leftrightarrow f$ is **concave down** at a (frowny)
- **Tangent slope** at a vs. **secant slope** $\frac{f(b)-f(a)}{b-a}$ over $[a, b]$

...

💡 Tip

Mean Value Theorem: somewhere in (a, b) the tangent slope equals the secant slope. So at $x = a$ the tangent can be smaller, equal, or bigger.

Exponential Function Basics

Property	e^x
Domain / Range	$\mathbb{R}$ / $(0, \infty)$
End behavior	$\to 0$ as $x \to -\infty$; $\to \infty$ as $x \to \infty$

...

Combined with chain & product rule:

...

$$\left(e^{g(x)}\right)' = e^{g(x)} \cdot g'(x), \qquad \left((x+p)e^x\right)' = e^x(x+p+1)$$

...

End behavior of $(x+p)e^x - q$:

- $x \to +\infty$: linear $\times$ exp blows up; $j \to +\infty$
- $x \to -\infty$: exponential decay wins; $(x+p)e^x \to 0$, so $j \to -q$

Reading Parameters From a Graph

Strategy:

1. Identify **readable points** (intercepts, asymptote levels, special values)
2. Each readable point gives **one equation** in the unknowns
3. Solve the resulting **system**, then round if instructed

...

Example with $j(x) = (x+p)e^x - q$:

...

- If $j(0) = -3$ (read from y-intercept), then $p - q = -3$.

...

- One equation, two unknowns; need a second point.

Inflection Points

Recipe:

1. Compute $j''(x)$
2. Solve $j''(x) = 0$
3. Verify sign change of j'' (or compute j''')

4. Compute y-coordinate $j(x_0)$

...

For $j(x) = (x + p)e^x - q$:

...

$$j'(x) = e^x(x + p + 1), \qquad j''(x) = e^x(x + p + 2)$$

...

Since $e^x > 0$ always: $j''(x) = 0 \Leftrightarrow x = -(p + 2)$.

Integration by Parts

$$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \, dx$$

...

Choosing: pick u so that u' is simpler; pick v' so it is easy to integrate.

...

Example: $\int (x + 1)e^x \, dx$

...

Pick $u = x + 1$, $v' = e^x$; then $u' = 1$, $v = e^x$:

...

$$\int (x + 1)e^x \, dx = (x + 1)e^x - \int e^x \, dx$$

...

$$(x + 1)e^x - e^x + C = x \cdot e^x + C$$

Concept Refresher - Probability - 15 Minutes

Contingency Tables

For two events A, B on a population of size N:

	A	$\bar{A}$	Sum
B	$N \cdot P(A \cap B)$	$N \cdot P(\bar{A} \cap B)$	$N \cdot P(B)$
$\bar{B}$	$N \cdot P(A \cap \bar{B})$	$N \cdot P(\bar{A} \cap \bar{B})$	$N \cdot P(\bar{B})$
Sum	$N \cdot P(A)$	$N \cdot P(\bar{A})$	N

Conditional Probability & Bayes

Definition: $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

...

Bayes' theorem:

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

...

> Tip
>
> On the exam, you can usually read $P(A \mid B)$ directly off the contingency table:
>
> $$P(A \mid B) = \frac{\text{count in } \ A \cap B}{\text{count in } \ B \ \text{ row/column}}$$

Binomial Distribution

Setting: n independent trials, each with success probability p.

...

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

...

Common variants:

- *"Exactly k"*: one term
- *"At most k"*: sum from 0 to k
- *"At least k"*: $1 - P(X \leq k-1)$

...

> ! Important
>
> Using the **complement** is often faster: e.g. $P(X \geq 1) = 1 - P(X = 0)$.

Coffee Break - 15 Minutes

Walkthrough: Function Analysis - 20 Minutes

The Problem

Consider $f(x) = 4x(x-3)^2 - 8$ and $g(x) = 4x - 8$.

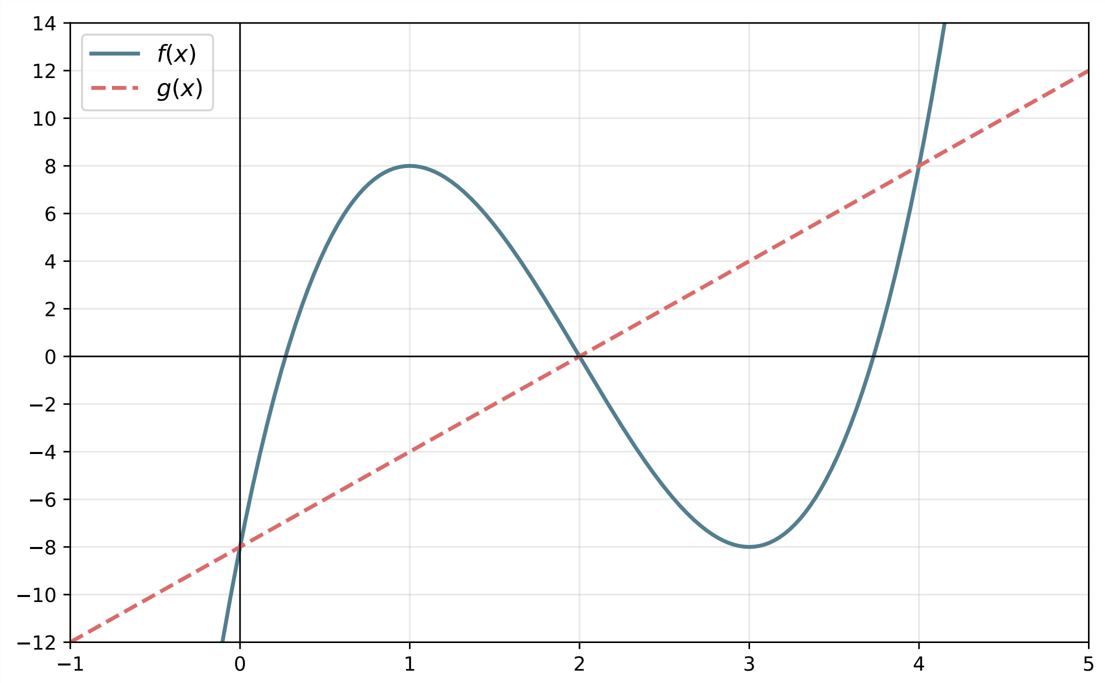

Figure 1: f(x) = 4x(x-3)^2 - 8 and g(x) = 4x - 8

Verify the Maximum at $x = 1$

Expand: $f(x) = 4x(x^2 - 6x + 9) - 8 = 4x^3 - 24x^2 + 36x - 8$

...

Step 1: $f'(1) = 0$?

...

$$f'(x) = 12x^2 - 48x + 36 = 12(x-1)(x-3) \qquad f'(1) = 0 \quad \checkmark$$

...

Step 2: $f''(1) < 0$?

...

$$f''(x) = 24x - 48 \qquad f''(1) = -24 < 0 \quad \checkmark$$

...

Step 3: maximum value: $f(1) = 4 \cdot 1 \cdot 4 - 8 = 8$

...

$$\boxed{\text{Local maximum at } (1, 8)}$$

Mirror Image at the y-Axis

$$f(x) = 4x(x^2 - 6x + 9) - 8$$

...

Replace x with $-x$:

...

$$f_1(x) = f(-x) = 4(-x)(-x-3)^2 - 8 = -4x(x+3)^2 - 8$$

...

Reasoning: every point $(a, f(a))$ becomes $(-a, f(a))$, mirroring the graph horizontally about the y-axis.

...

$$\boxed{f_1(x) = -4x(x+3)^2 - 8}$$

Find the Intersections

Set $f(x) = g(x)$:

...

$$4x(x-3)^2 - 8 = 4x - 8$$

$$4x(x-3)^2 - 4x = 0$$

$$4x\left[(x-3)^2 - 1\right] = 0$$

...

$$4x(x^2 - 6x + 8) = 0 \quad \Rightarrow \quad 4x(x-2)(x-4) = 0$$

...

Intersections: $x = 0, \quad x = 2, \quad x = 4$

Show the Two Areas Are Equal

Difference: $h(x) = f(x) - g(x) = 4x(x-2)(x-4) = 4x^3 - 24x^2 + 32x$

...

Sign check: at $x = 1$: $h(1) = 4 \cdot 1 \cdot (-1)(-3) = 12 > 0$ (f above g)

...

Region 1 ($0 \le x \le 2$):

...

$$A_1 = \int_0^2 h(x)\,dx = \left[x^4 - 8x^3 + 16x^2\right]_0^2 = 16 - 64 + 64 = 16$$

...

Region 2 ($2 \le x \le 4$):

...

$$A_2 = \left|\left[x^4 - 8x^3 + 16x^2\right]_2^4\right| = |(256 - 512 + 256) - 16| = 16$$

...

$$\boxed{A_1 = A_2 = 16 \quad \checkmark}$$

Walkthrough: Exponential Function - 15 Minutes

The Exponential Problem

Consider $j(x) = (x + p) \cdot e^x - q$ with parameters p, q shown below:

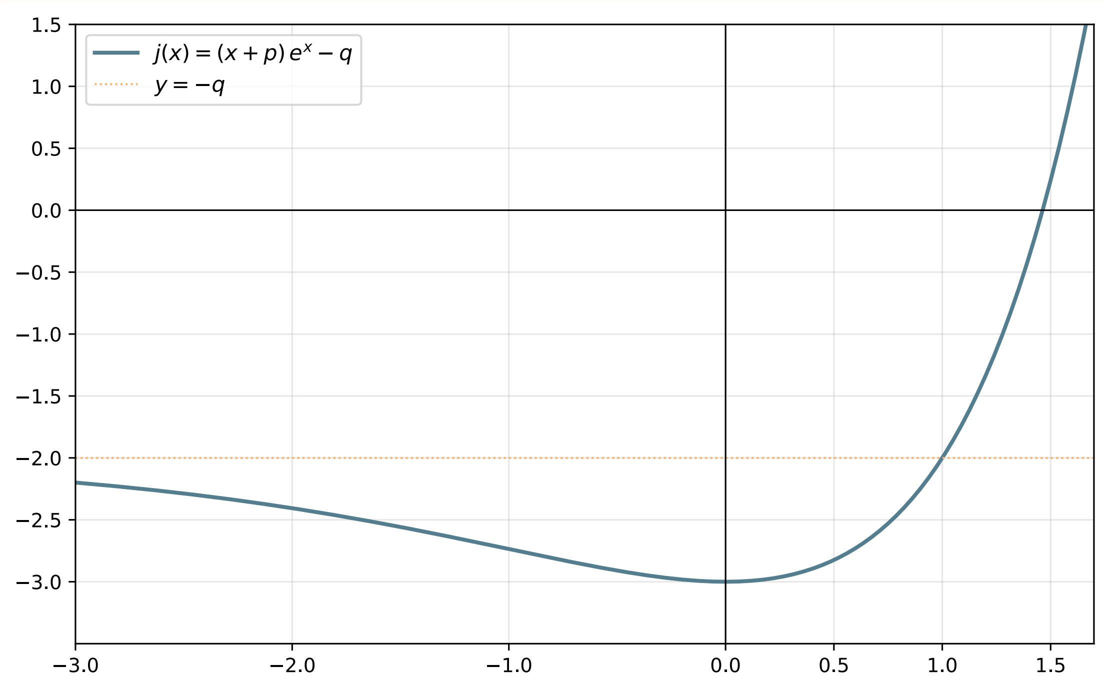

Figure 2: Graph of j(x) = (x + p) eˣ − q

Read p and q From the Graph

Two readable points:

- y-intercept: $j(0) \approx -3$
- x-intercept: $j(1.5) \approx 0$

...

System:

$$j(0) = p - q = -3 \quad (I)$$

$$j(1.5) = (1.5 + p)e^{1.5} - q = 0 \quad (II)$$

...

Round to one decimal: $\boxed{p \approx -1, \quad q \approx 2}$

End Behavior

For $j(x) = (x-1)e^x - 2$:

...

As $x \to +\infty$: $(x-1) \to \infty$ and $e^x \to \infty$:

$$\lim_{x \to +\infty} j(x) = +\infty$$

...

As $x \to -\infty$: exponential decay beats linear growth, so $(x-1)e^x \to 0$:

$$\lim_{x \to -\infty} j(x) = 0 - 2 = -2$$

...

$$\boxed{j \to +\infty \text{ at } +\infty; \quad y = -2 \text{ horizontal asymptote at } -\infty}$$

Inflection Point

For $j(x) = (x-1)e^x - 2$, **compute:**

...

$$j'(x) = e^x + (x-1)e^x = e^x \cdot x \qquad j''(x) = e^x + x \cdot e^x = e^x(x+1)$$

...

Set $j''(x) = 0$: since $e^x > 0$ always, $x + 1 = 0 \Rightarrow x = -1$.

...

y-coordinate:

$$j(-1) = (-1-1)e^{-1} - 2 = -\frac{2}{e} - 2 \approx -2.74$$

...

$$\boxed{\text{Inflection point at } \left(-1, -\frac{2}{e} - 2\right) \approx (-1, -2.74)}$$

The Scenario

A fitness studio offers a workout-tracking app to its members.

...

Given:

- 20% of random people are members (M): $P(M) = 0.20$
- 75% of members use the app (A): $P(A \mid M) = 0.75$

- 10% of non-members still use such an app: $P(A \mid \bar{M}) = 0.10$

Build the Contingency Table

Use $N = 100,000$ people:

- M: $100,000 \times 0.20 = 20,000$; $\bar{M}$: $80,000$
- $M \cap A$: $20,000 \times 0.75 = 15,000$; $M \cap \bar{A}$: $5,000$
- $\bar{M} \cap A$: $80,000 \times 0.10 = 8,000$; $\bar{M} \cap \bar{A}$: $72,000$

. . .

	M	$\bar{M}$	Sum
A	15,000	8,000	**23,000**
$\bar{A}$	5,000	72,000	**77,000**
Sum	**20,000**	**80,000**	**100,000**

Bayes: $P(M \mid A)$

A person uses the app. What is the probability they are a member?

. . .

Read directly from the table:

$$P(M \mid A) = \frac{|M \cap A|}{|A|} = \frac{15,000}{23,000} \approx 0.6522$$

. . .

$$\boxed{P(M \mid A) \approx 65.2\%}$$

. . .

> Tip
>
> Even though only 20% of people are members, having the app raises the probability of membership to 65%, because the app is much more common among members.

Binomial: Sample of $n = 8$

Use $p = P(M) = 0.20$.

. . .

Exactly 2 are members:

. . .

$$P(X = 2) = \binom{8}{2}(0.2)^2(0.8)^6 = 28 \cdot 0.04 \cdot 0.2621 \approx 0.2936$$

...

At most 2 are members:

...

- $P(X = 0) = (0.8)^8 \approx 0.1678$, $P(X = 1) = 8 \cdot 0.2 \cdot (0.8)^7 \approx 0.3355$
- $P(X \leq 2) \approx 0.1678 + 0.3355 + 0.2936 \approx 0.7969 \approx 79.7\%$

Binomial: Range Probability

Between 2 and 4 members (inclusive):

- $P(X = 2) \approx 0.2936$
- $P(X = 3) = \binom{8}{3}(0.2)^3(0.8)^5 = 56 \cdot 0.008 \cdot 0.3277 \approx 0.1468$
- $P(X = 4) = \binom{8}{4}(0.2)^4(0.8)^4 = 70 \cdot 0.0016 \cdot 0.4096 \approx 0.0459$

...

$$\boxed{P(2 \leq X \leq 4) \approx 0.4863 \approx 48.6\%}$$

Exam Tips & Key Takeaways - 5 Minutes

What to Remember

- **Verify extrema in 3 steps:** $f'(a) = 0$, sign of $f''(a)$, value $f(a)$
- **Areas between curves:** find intersections; integrate the absolute difference; split at sign changes
- **Reading parameters:** every readable point gives one equation; solve the system, then round
- **Contingency tables** turn conditional info into counts
- **Binomial:** identify n, p, and the type of event (exact, at most, at least, range)

...

> Tip
>
> The **tasks file** for 09-07 has another exam-style problem set covering the same techniques with different numbers.