Session 09-05 - Final Recap: Calculus & Curve Sketching

Section 09: Exam Preparation

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Concept Refresher - Part A - 25 Minutes

Domain & Range

Function type	Domain rule	Range
Polynomial $p(x)$	$\mathbb{R}$	Depends on degree & leading coeff.
Rational $\frac{p(x)}{q(x)}$	$\mathbb{R} \setminus \{x : q(x) = 0\}$	Excludes horizontal asymptote value
$\ln(g(x))$	$g(x) > 0$	$\mathbb{R}$
$\sqrt{g(x)}$	$g(x) \geq 0$	$[0, \infty)$
$e^{g(x)}$	$\mathbb{R}$	$(0, \infty)$

...

> Tip
>
> Key exam skill: Always check for division by zero, negative log/sqrt arguments, and context restrictions ($t \geq 0$ for time).

Limits & Asymptotes

Horizontal asymptotes of $\frac{a_n x^n + \ldots}{b_m x^m + \ldots}$; compare the degrees:

- $n < m$ (numerator degree smaller): HA is $y = 0$
- $n = m$ (degrees equal): HA is $y = \frac{a_n}{b_m}$
- $n > m$ (numerator degree larger): no horizontal asymptote

...

Vertical: denominator zeros after cancelling common factors.

- Cancelled factors $\rightarrow$ holes, not asymptotes!

Limits & Asymptotes: Example

Example: Find all asymptotes of $f(x) = \frac{2x^2-8}{x^2-x-6}$.

. . .

Step 1, factor: $f(x) = \frac{2(x-2)(x+2)}{(x-3)(x+2)} = \frac{2(x-2)}{x-3}, \quad x \neq -2$

- Hole at $x = -2$ (cancelled factor), with $y = \frac{2(-4)}{-5} = \frac{8}{5}$
- Vertical asymptote at $x = 3$ (remaining denominator zero)
- Horizontal asymptote $y = 2$ (degrees equal, ratio $\frac{2}{1} = 2$)

End Behavior of Polynomials

The leading term $a_n x^n$ determines what happens as $x \to \pm\infty$:

- Even degree, $a_n > 0$: ↑ both ends | $a_n < 0$: ↓ both ends
- Odd degree, $a_n > 0$: ↓ left, ↑ right | $a_n < 0$: ↑ left, ↓ right

. . .

Example 1: $f(x) = -3x^4 + 5x^2 - 1$

. . .

Leading term: $-3x^4 \quad$ (even degree, $a_n < 0$) $\quad \Rightarrow \quad$ ↓ both ends

. . .

Example 2: $f(x) = \frac{1}{8}x^3 - \frac{3}{4}x^2 + 4$

. . .

Leading term: $\frac{1}{8}x^3 \quad$ (odd degree, $a_n > 0$) $\quad \Rightarrow \quad$ ↓ left, ↑ right

. . .

> Tip
>
> Even degree = same direction on both sides. Odd degree = opposite directions.

Evaluating Limits

Rational functions as $x \to \infty$: divide every term by the highest power of x in the denominator.

$$\lim_{x \to \infty} \frac{3x^2 + 1}{x^2 - x}$$

. . .

$$\lim_{x \to \infty} \frac{3 + 1/x^2}{1 - 1/x}$$

. . .

$$\frac{3}{1} = 3$$

Derivative Rules

Product rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$

...

$$\text{Example:} \quad (x^2 \ln x)' = 2x \cdot \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x$$

...

Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

...

$$\text{Example:} \quad \left(\frac{x^2}{x+1}\right)' = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$$

...

Chain rule: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

...

$$\text{Example:} \quad (e^{x^2 - 3x})' = e^{x^2 - 3x} \cdot (2x - 3)$$

Common Derivative Pitfalls

- Forgetting the chain rule: $(e^{g(x)})' = e^{g(x)} \cdot g'(x)$, not just $e^{g(x)}$!
- Product rule has two terms: $(f \cdot g)' \neq f' \cdot g'$
- Sign in quotient rule: it's $f'g - fg'$ in the numerator; order matters!
- $\frac{d}{dx} \ln(g(x)) = \frac{g'(x)}{g(x)}$: don't forget the $g'(x)$ in the numerator

Trigonometric Derivatives

Property	$\sin(x)$	$\cos(x)$
Domain / Range	$\mathbb{R}$ / $[-1, 1]$	$\mathbb{R}$ / $[-1, 1]$
Period	2π	2π
Derivative	$\cos(x)$	$-\sin(x)$

...

With chain rule:

...

$(\sin(kx))' = k \cos(kx), \quad (\cos(kx))' = -k \sin(kx)$

...

Angle of intersection:

...

A line with slope m meets the x-axis at angle $\alpha = \arctan(m)$.

...

> Tip
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> Make sure your calculator is in degree mode if the problem asks for degrees!

Tangent Lines & Curve Sketching

Tangent line at $x = a$: $\boxed{t(x) = f'(a)(x - a) + f(a)}$

...

1. Domain: where is f defined?
2. Zeros & y-intercept: $f(x) = 0$; $f(0)$
3. $f'(x) = 0$: critical points $\rightarrow$ local max/min
4. $f''(x) = 0$: inflection points, concavity
5. Asymptotes / end behavior
6. Sketch: combine all information

...

> ! Important
>
> Even if only asked for the sketch, showing steps 1-5 earns partial credit!

Antiderivatives

Key rules:

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- $\int \frac{1}{x} \, dx = \ln|\, x \,| + C$
- $\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C$
- $\int \sin(x) \, dx = -\cos(x) + C, \quad \int \cos(x) \, dx = \sin(x) + C$

...

> ! Important
>
> Never forget $+C$ on indefinite integrals!

Graphical Differentiation & Integration

Reading f' from f (graphical differentiation):

- f has local max/min $\Rightarrow f'(x) = 0$
- f is increasing $\Rightarrow f' > 0$ | f decreasing $\Rightarrow f' < 0$
- If f is degree n, then f' is degree $n - 1$

...

Reading $\boldsymbol{F}$ from $\boldsymbol{f}$ (graphical integration):

- $f > 0 \Rightarrow F$ increasing | $f < 0 \Rightarrow F$ decreasing
- f changes sign $\Rightarrow F$ has a local extremum
- $f = 0$ but no sign change (double root) $\Rightarrow F$ has an inflection point
- If f is degree n, then F is degree $n + 1$

Coffee Break - 15 Minutes

Exam Problem Walkthrough - 45 Minutes

The Problem

Consider the following third degree polynomial $f(x)$:

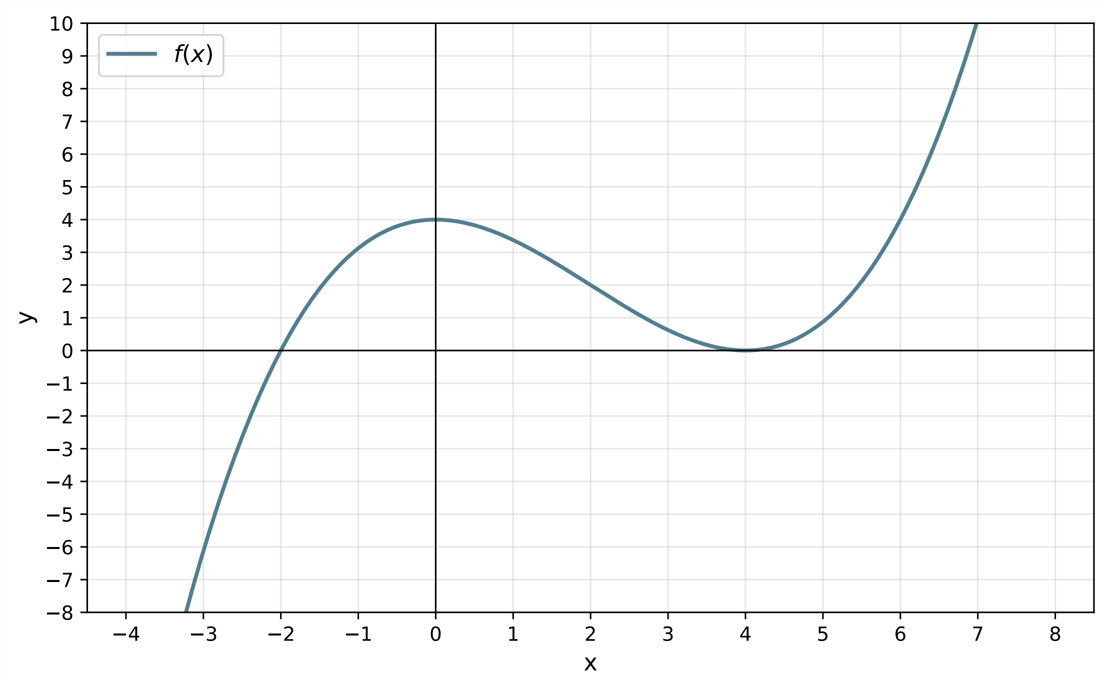

Figure 1: Graph of the third degree polynomial f(x)

What Can We Read from This Graph?

- Zeros: $f(-2) = 0$ and $f(4) = 0$
- At $x = -2$: the graph crosses the x-axis $\Rightarrow$ single root
- At $x = 4$: graph touches the x-axis and turns around $\Rightarrow$ double root
- Local maximum: at $x = 0$ with $f(0) = 4$
- f is increasing for $x < 0$, decreasing for $0 < x < 4$, increasing for $x > 4$

...

> Tip
>
> Single vs. double root: If the graph crosses the x-axis, it's a single root. If it touches and turns, it's a double root (the factor appears squared).

Part a) Graph the Derivative $f'(x)$

Where is $f'(x) = 0$? At every local max/min and every double-root:

- $f'(0) = 0$ (local maximum at $x = 0$)
- $f'(4) = 0$ (double root; the tangent is horizontal there)

...

What is the sign of f'?

- f increasing for $x < 0 \Rightarrow f' > 0$
- f decreasing for $0 < x < 4 \Rightarrow f' < 0$
- f increasing for $x > 4 \Rightarrow f' > 0$
- f is cubic (degree 3) with positive leading coefficient $\Rightarrow f'$ is a parabola opening upward

Part a) Solution

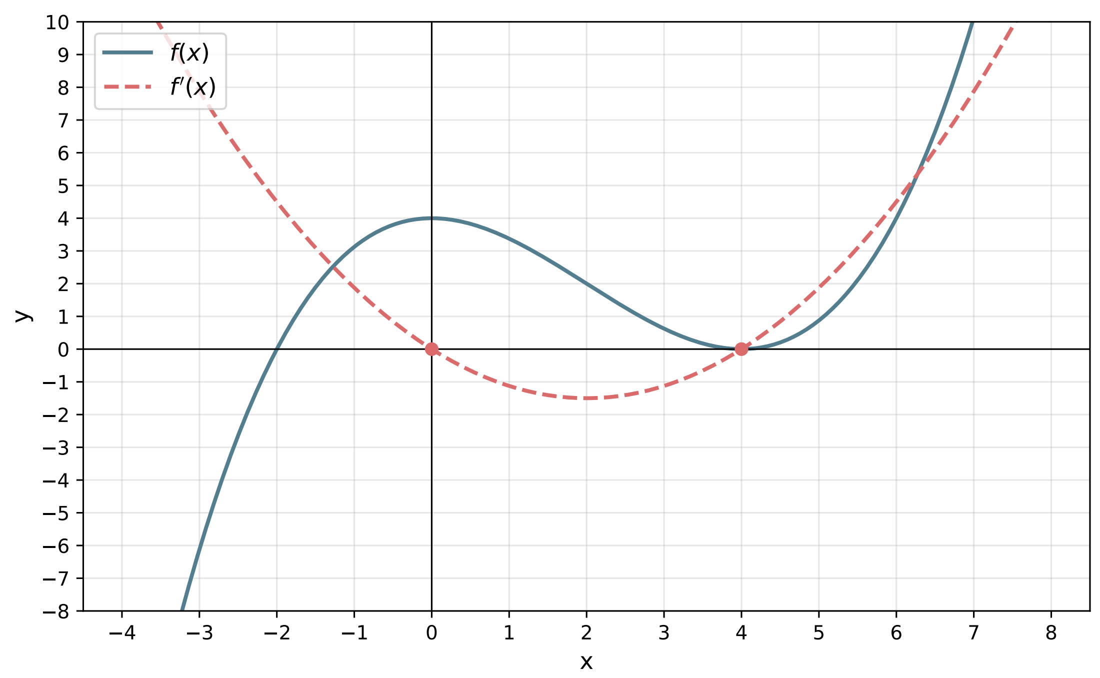

Figure 2: Graph of f(x) with its derivative f'(x)

Part b) Graph the Antiderivative $F(x)$

Strategy: F increases where $f > 0$ and decreases where $f < 0$.

- $f < 0$ for $x < -2 \Rightarrow F$ is decreasing
- $f > 0$ for $-2 < x < 4$ and $x > 4 \Rightarrow F$ is increasing
- $f(-2) = 0$ and f changes sign $\Rightarrow F$ has a local minimum at $x = -2$
- $f(4) = 0$ but f does not change sign (double root) $\Rightarrow F$ has an inflection point at $x = 4$
- f is degree 3 $\Rightarrow F$ is degree 4 (grows much faster, so we scale F by $\frac{1}{4}$ to fit the plot)

Part b) Solution

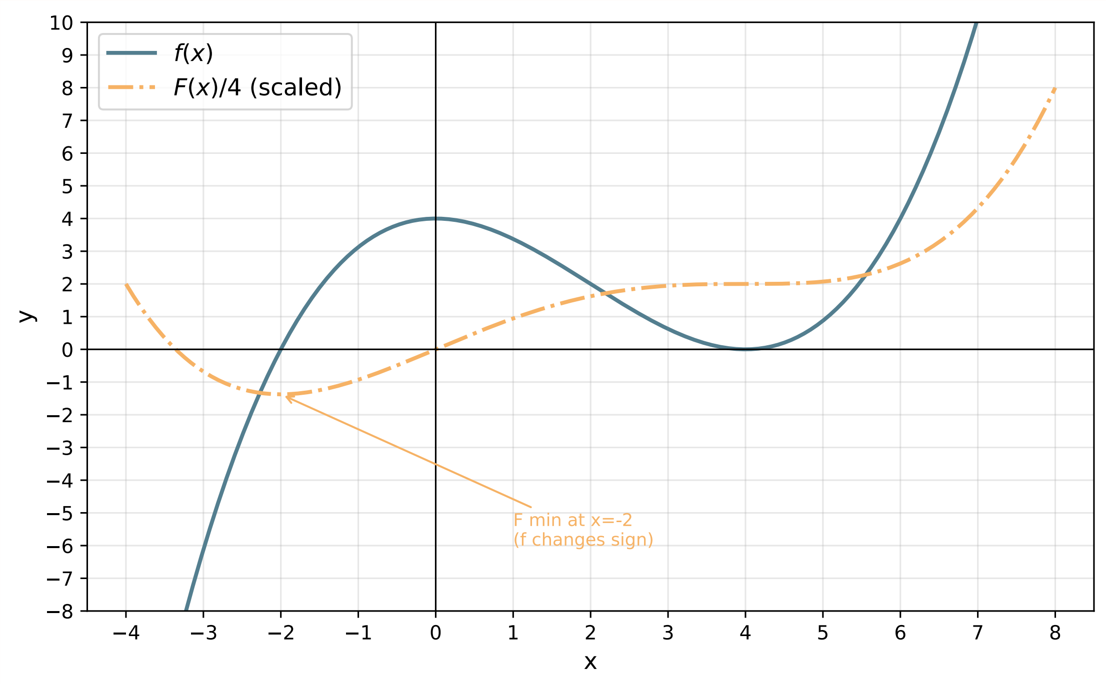

Figure 3: Graph of f(x) with its antiderivative F(x) (scaled by 1/4 for visibility)

Part c) Strategy: Setting Up the System

We want $f(x) = ax^3 + bx^2 + cx + d$: that's 4 unknowns, so we need 4 equations.

...

What can we read from the graph?

- $f(0) = 4$ (the y-intercept) $\quad\Rightarrow\quad d = 4$
- $f'(0) = 0$ (local max; horizontal tangent) $\quad\Rightarrow\quad c = 0$
- $f(-2) = 0$ (zero of the function)
- $f(4) = 0$ (zero of the function)

...

> Tip
>
> Look for points where you can read exact values: zeros, y-intercept, and extrema give the cleanest equations.

Part c) Solving the System

With $c = 0$ and $d = 4$, the two remaining equations are:

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$$f(-2) = 0: \quad -8a + 4b + 4 = 0 \quad \Rightarrow \quad 2a - b = 1 \quad (I)$$

...

$f(4) = 0: \quad 64a + 16b + 4 = 0 \quad \Rightarrow \quad 64a + 16b = -4 \quad (II)$ $

...

$$\boxed{f(x) = \frac{1}{8}x^3 - \frac{3}{4}x^2 + 4 = \frac{1}{8}(x+2)(x-4)^2}$$

Part d) Tangent Line at $x = 6$

Step 1: function value at the point

...

- $f(6) = \frac{216}{8} - \frac{108}{4} + 4 = 27 - 27 + 4 = 4$

...

Step 2: slope (derivative at the point)

...

- $f'(x) = \frac{3x^2}{8} - \frac{3x}{2} \qquad f'(6) = \frac{108}{8} - 9 = \frac{27}{2} - 9 = \frac{9}{2}$

...

Step 3: plug into tangent formula $t(x) = f'(a)(x - a) + f(a)$

...

- $t(x) = \frac{9}{2}(x - 6) + 4 = \frac{9}{2}x - 27 + 4 = \boxed{\frac{9}{2}x - 23}$

Part e) Angle of Intersection

The tangent has slope $m = \frac{9}{2} = 4.5$.

...

The angle between the tangent and the x-axis:

...

$$\tan(\alpha) = m = 4.5 \qquad \Rightarrow \qquad \alpha = \arctan(4.5) \approx 77.47°$$

...

$$\boxed{\alpha \approx 77.5°}$$

Visual

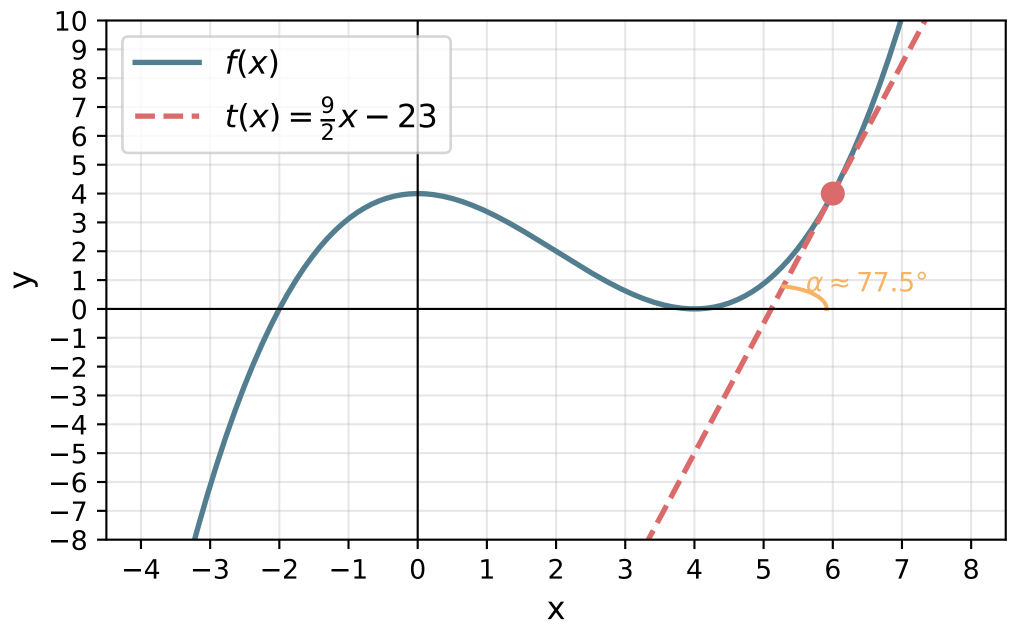

Figure 4: Graph of f(x) with tangent line at x = 6 and angle of intersection

Exam Tips & Key Takeaways - 5 Minutes

What to Remember

- Read the graph carefully before calculating; identify zeros, extrema, special points
- 4 unknowns need 4 equations: look for readable values (zeros, intercepts, extrema)
- Show your work on every step; partial credit is awarded for correct reasoning
- Verify your results: plug solutions back into the original equations
- Don't forget the chain rule: the #1 source of derivative errors
- $+C$ on indefinite integrals: easy points lost if forgotten
- Check your calculator mode: degrees vs. radians for $\arctan$

...

> Tip
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> Practice with the tasks file and the mock exams, same problem structure, different numbers!