

Session 09-03 - Integral Calculus Review

Section 09: Exam Preparation

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Warm-Up Quiz - 10 Minutes

Quick Review Questions

Work individually for 5 minutes, then discuss with the class (5 minutes)

1. Find the antiderivative: $\int (4x^3 - 2x + 5) dx$
2. Rewrite $\frac{3}{\sqrt{x}}$ using a negative exponent, then integrate.
3. Evaluate: $\int_0^1 e^{2x} dx$
4. A marginal cost function is $MC(x) = 6x + 20$. If fixed costs are 500, find the total cost function $C(x)$.

Guided Practice - Part A - 25 Minutes

Practice Problem 1

Antiderivative Refresh (Sections 06 & 01)

Work in pairs

Find the antiderivatives of each expression. Rewrite first using exponent rules where needed!

- (a) $3x^{-2}$
- (b) e^{5x}
- (c) $\frac{1}{x}$
- (d) $4\sqrt{x}$ (rewrite as $4x^{1/2}$)
- (e) $\frac{2}{x^3}$ (rewrite as $2x^{-3}$)

Practice Problem 2

Definite Integral Computation (Section 06)

Work in pairs

Evaluate each definite integral:

- (a) $\int_0^2 (x^2 - 1) dx$

(b) $\int_1^e \frac{3}{x} dx$

(c) $\int_0^1 2e^{3x} dx$

(d) $\int_{-1}^1 (x^3 - x) dx$ (exploit symmetry!)

Practice Problem 3

Continuous Compounding (Sections 08 & 06)

Work in pairs

An income stream pays $f(t) = 1000$ euro/year continuously, and the discount rate is $r = 0.05$.

The present value over $T = 10$ years is:

$$PV = \int_0^{10} 1000 \cdot e^{-0.05t} dt$$

(a) Compute this integral.

(b) Interpret the result: what does the present value tell us?

Practice Problem 4

Area Between Curves (Sections 06 & 02)

Work in pairs

Find the area enclosed by the curves:

$$f(x) = x^2 \quad \text{and} \quad g(x) = 2x + 3$$

(a) Find the intersection points (use the quadratic formula from Sec 02).

(b) Determine which function is on top in the enclosed region.

(c) Set up and compute the integral for the enclosed area.

Practice Problem 5

Consumer & Producer Surplus (Sections 06 & 08)

Work in pairs

A market has demand $D(q) = 100 - 2q^2$ and supply $S(q) = 10 + q^2$.

(a) Find the equilibrium price and quantity.

(b) Calculate consumer surplus.

(c) Calculate producer surplus.

(d) Interpret the results.

Guided Practice - Part B - 30 Minutes

Practice Problem 6

Substitution Method (Section 06)

Work in pairs

Evaluate each integral using substitution:

(a) $\int 2x \cdot e^{x^2} dx$

(b) $\int \frac{3x}{x^2+4} dx$

(c) $\int x(x^2 + 1)^4 dx$

Practice Problem 7

Average Cost Word Problem (Sections 06 & 08)

Work in pairs

A company's daily production cost rate is $C(t) = 200 + 50e^{-0.1t}$ euro/hour over an 8-hour shift ($0 \leq t \leq 8$).

(a) Find the total production cost over the 8-hour shift: $\int_0^8 (200 + 50e^{-0.1t}) dt$

(b) Find the average hourly cost: $\bar{C} = \frac{1}{8} \int_0^8 C(t) dt$

(c) At what time t does $C(t)$ equal the average hourly cost?

Practice Problem 8

Area Between Exponential and Linear (Sections 06 & 01)

Work in pairs

Consider the functions $f(x) = e^x$ and $g(x) = x + 1$ on the interval $[0, 1]$.

(a) Which function is on top on $[0, 1]$?

(b) Find the area enclosed between $f(x)$ and $g(x)$ on $[0, 1]$.

Coffee Break - 15 Minutes

Collaborative Problem-Solving - Part A - 25 Minutes

Practice Problem 9

Integration by Parts (Section 06)

Think individually (2 min), then work in groups of 3-4

Evaluate the following integrals using integration by parts:

(a) $\int x \cdot e^{-2x} dx$

(b) $\int x^2 \cdot \ln(x) dx$

Practice Problem 10

Factory Cost Reconstruction (Sections 06 & 08)

Think individually (2 min), then work in groups of 3-4

A factory's marginal cost function is:

$$MC(x) = 0.06x^2 - 3x + 80$$

and fixed costs are 5000 euro.

- Find the total cost function $C(x)$.
- Find the average cost function $\bar{C}(x)$.
- Find the production level that minimizes average cost.
- Show that at this level, marginal cost equals average cost.

Practice Problem 11

Pollution Accumulation Model (Section 06)

Think individually (2 min), then work in groups of 3-4

A factory releases pollutants at a rate of:

$$P(t) = 50 \cdot e^{-0.1t} \text{ tons/year}$$

- How much pollution is released in the first 5 years?
- How much pollution is released in the long run ($t \rightarrow \infty$)?
- After how many years has 90% of the total long-run pollution been released?

Coffee Break - 10 Minutes

Collaborative Problem-Solving - Part B - 25 Minutes

Practice Problem 12

Long-Run Accumulation (Section 06)

Think individually (2 min), then work in groups of 3-4

A machine generates waste at a rate of $W(t) = 80 \cdot e^{-0.25t}$ kg/year.

- Find the total waste produced in the first T years: $Q(T) = \int_0^T 80e^{-0.25t} dt$
- Compute $Q(10)$, $Q(20)$, and $Q(50)$.
- What value does $Q(T)$ approach as T becomes very large?
- After how many years has 95% of the long-run total been produced? (Use logarithms.)

Practice Problem 13

Break-Even Analysis via Integration (Sections 06 & 08)

Think individually (2 min), then work in groups of 3-4

A company launches a new product. The profit rate is $P(t) = 6t - t^2 - 5 = -(t - 1)(t - 5)$ thousand euro/month, where t is months since launch.

- Which months is the company making a profit? (Find where $P(t) > 0$.)
- Find the total profit over the profitable period.
- First month, the company operates at a loss. Total loss $\int_0^1 P(t) dt$?
- Find the net total outcome over the first 8 months: $\int_0^8 P(t) dt$.

Practice Problem 14

Total Profit from Marginal Analysis (Sections 06, 05 & 08)

Think individually (2 min), then work in groups of 3-4

A firm has:

- Marginal revenue: $MR(x) = 100 - 0.5x$
- Marginal cost: $MC(x) = 20 + 0.3x$

- At what quantity x^* does $MR = MC$?
- Compute total profit: $\int_0^{x^*} [MR(x) - MC(x)] dx$.
- Verify with $R(0) = 0$ and $C(0) = 500$.

Wrap-Up & Next Steps

Key Takeaways

- Antiderivatives & power rule, rewrite roots and fractions before integrating
- Definite integrals, give net signed area; use odd/even symmetry to simplify
- Substitution and IBP, match inner-function patterns or apply LIATE
- Economic applications, surplus, present value, and marginal-to-total recovery
- Improper integrals, exponential decay converges to finite long-run totals