Session 09-02 - Differential Calculus Review

Section 09: Exam Preparation

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Warm-Up Quiz - 10 Minutes

Quick Review Questions

Work individually for 5 minutes, then discuss with the class (5 minutes)

1. Evaluate the limit $\lim_{x\to 0} \frac{e^{2x}-1}{x}$.
2. Find $f'(x)$ for $f(x) = x^3 \cdot e^{-x}$.
3. Solve $3x^2 - 12x + 9 = 0$.
4. For $f(x) = x^2 - 6x + 10$, find the vertex and classify it as min or max using the derivative.

Guided Practice - Part A - 25 Minutes

Practice Problem 1

e-Limits and Special Limits (Sections 05)

Work in pairs

Evaluate each limit:

(a) $\lim_{x\to 0} \frac{e^{3x}-1}{x}$

(b) $\lim_{n\to \infty} \left(1 + \frac{2}{n}\right)^n$

(c) $\lim_{x\to \infty} \frac{3x^2+1}{x^2-x}$

(d) $\lim_{x\to 0} \frac{\ln(1+x)}{x}$

Practice Problem 2

Differentiation Rules Drill (Sections 05)

Work in pairs

Find the derivatives:

(a) Product rule: $f(x) = x^2 \cdot \ln(x)$

(b) Chain rule: $g(x) = e^{x^2-3x}$

(c) Quotient + Chain: $h(x) = \frac{(2x+1)^3}{x-1}$

Practice Problem 3

Fencing Optimization (Sections 03 & 05)

Work in pairs

A farmer has 200 meters of fencing to enclose a rectangular area along a straight river (no fencing needed on the river side). What dimensions maximize the enclosed area? Let x be the side perpendicular to the river and express the area as a function of x alone, then optimize.

Practice Problem 4

Complete Curve Sketch (Sections 05)

Work in pairs

Perform a complete curve analysis of:

$$f(x) = (x^2 - 4) \cdot e^{-x}$$

(a) Domain

(b) Zeros (factorization!)

(c) Behavior as $x \to \pm\infty$ (e-limits!)

(d) $f'(x)$ (product rule) and critical points

(e) $f''(x)$ and inflection points

(f) Sketch

Practice Problem 5

Funktionsscharen (Sections 05)

Work in pairs

For $f_a(x) = x^3 - 3ax^2 + 4a$ with parameter $a \in \mathbb{R}$:

(a) Find all extrema as a function of the parameter a.

(b) Find the value of a for which a local extremum lies on the x-axis.

(c) Sketch $f_1(x)$ and $f_2(x)$ on the same axes.

Guided Practice - Part B - 30 Minutes

Practice Problem 6

Derivative of Logarithmic Function (Sections 05 & 01)

Work in pairs

For the function $f(x) = \ln(x^2 + 1)$:

(a) Determine the domain of f.

(b) Find $f'(x)$ using the chain rule.

(c) Find all critical points and classify them.

(d) Determine $f''(x)$ and discuss concavity.

(e) Sketch the graph.

Practice Problem 7

Exponential Rate of Change (Sections 05 & 08)

Work in pairs

A bank account grows according to $A(t) = 5000 \cdot e^{0.04t}$, where A is in euros and t is in years.

(a) Find the rate of change $A'(t)$.

(b) At what rate is the account growing after 10 years?

(c) How long until the account doubles?

(d) Show that the rate of change is proportional to the current balance.

Practice Problem 8

Complete Curve Sketch of Rational Function (Sections 05 & 04)

Perform a complete curve analysis of:

$$f(x) = \frac{x}{x^2 + 1}$$

(a) Domain

(b) Symmetry

(c) Zeros and y-intercept

(d) Asymptotes (vertical? horizontal?)

(e) $f'(x)$ and critical points (extrema)

(f) $f''(x)$ and inflection points

Coffee Break - 15 Minutes

Collaborative Problem-Solving - Part A - 25 Minutes

Practice Problem 9

Business Profit Optimization (Sections 05 & 03)

Think individually (2 min), then work in groups of 3-4

A company's revenue and cost functions are:

$$R(x) = 1000x \cdot e^{-0.01x}, \qquad C(x) = 500 + 8x$$

where x is units produced.

(a) Find the profit function $P(x)$.

(b) Find the production level that maximizes profit.

(c) Confirm with the second derivative test.

(d) Calculate the maximum profit.

Practice Problem 10

Oblique Asymptote & Full Sketch (Sections 05 & 04)

Think individually (2 min), then work in groups of 3-4

$$f(x) = \frac{x^2 + 3x}{x - 1}$$

(a) Perform polynomial long division.

(b) Identify the oblique asymptote and vertical asymptote.

(c) Find zeros and y-intercept.

(d) Use derivatives to find extrema.

(e) Complete the sketch showing the oblique asymptote.

Practice Problem 11

Tangent Line Challenge (Sections 02 & 05)

Think individually (2 min), then work in groups of 3-4

Find the equation of all tangent lines to $f(x) = x^3 - 3x$ that pass through the point $(0, 2)$. A tangent line at $x = a$ has the form $y = f'(a)(x - a) + f(a)$. Require that this line passes through $(0, 2)$ and solve for a.

Coffee Break - 10 Minutes

Collaborative Problem-Solving - Part B - 25 Minutes

Practice Problem 12

Newton's Cooling Law (Sections 05 & 04)

Think individually (2 min), then work in groups of 3-4

A cup of coffee cools according to $T(t) = 20 + 60e^{-0.1t}$ (temperature in degrees C, time t in minutes).

(a) What is the initial temperature of the coffee?

(b) What is the room temperature (asymptote)?

(c) Find $T'(t)$ and the rate of cooling at $t = 5$ minutes.

(d) When does the coffee reach 40 degrees C?

(e) Show that $T'(t) = -0.1(T(t) - 20)$ — the rate of cooling is proportional to the temperature difference.

Practice Problem 13

Limits by Algebraic Manipulation (Sections 05)

Think individually (2 min), then work in groups of 3-4

Evaluate the following limits using algebraic techniques (factoring, expanding, rationalizing):

(a) $\lim_{x\to 3} \frac{x^2-9}{x^2-5x+6}$ (factor numerator and denominator)

(b) $\lim_{x\to\infty} \frac{5x^3-2x}{3x^3+x^2-1}$ (divide by highest power)

(c) $\lim_{x\to 0} \frac{e^{2x}-1}{e^x-1}$ (factor using $e^{2x} - 1 = (e^x - 1)(e^x + 1)$)

Practice Problem 14

Funktionsschar: Common Points and Tangent Lines (Sections 05)

Think individually (2 min), then work in groups of 3-4

For the family of functions $f_t(x) = x^2 - 2tx + t^2 + t$ with $t \in \mathbb{R}$:

(a) Show that $f_t(x) = (x - t)^2 + t$.

(b) Find the vertex of f_t as a function of t.

(c) Find the value(s) of t for which the vertex lies on the x-axis (i.e., f_t has a double zero).

(d) Find the tangent line to $f_1(x)$ at $x = 3$ and to $f_2(x)$ at $x = 3$. Do these tangent lines intersect? If so, where?

Wrap-Up & Next Steps

Key Takeaways

- Differentiation rules (product, chain, quotient) are the core toolkit for curve analysis and optimization
- Complete curve sketches require systematic analysis: domain, symmetry, zeros, asymptotes, derivatives, and inflection points
- Exponential and logarithmic functions appear throughout business applications — mastering their derivatives and limits is essential
- Funktionsscharen (function families) test your ability to work with parameters and generalize results
- Algebraic manipulation (factoring, long division, substitution) is key to evaluating limits and simplifying derivatives