Session 08-02 - Annuities & Loan Amortization

Section 08: Financial Mathematics

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Entry Quiz - 10 Minutes

Quick Review from Session 08-01

Test your understanding of Compound Interest

1. Find the 6th term of the geometric sequence: $4, 12, 36, \ldots$
2. Calculate the future value of 5,000 invested at 6% compounded quarterly for 5 years.
3. Find the effective annual rate for 8% compounded monthly.
4. How much should you invest today at 5% annual interest to have 15,000 in 8 years?

Learning Objectives

What You'll Master Today

- Understand ordinary annuities and annuities due
- Calculate future value of regular payment streams
- Calculate present value of annuity payments
- Solve for payment amounts and number of payments
- Build loan amortization schedules
- Analyze interest vs. principal in loan payments

Part A: Introduction to Annuities

What is an Annuity?

An annuity is a series of equal payments made at regular intervals.

. . .

Examples:

- Monthly rent payments
- Salary deposits
- Loan repayments
- Retirement contributions
- Insurance premiums

...

ℹ Note

The key: Equal payments at equal intervals

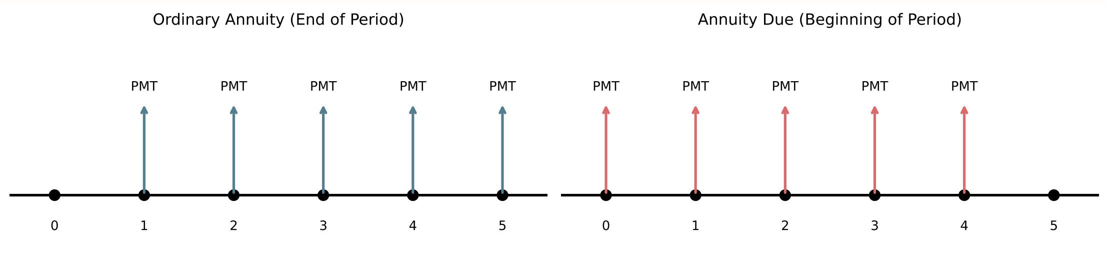
Types of Annuities

! Two Main Types

Ordinary Annuity (Nachschussig): Payments made at the end of each period

Annuity Due (Vorschussig): Payments made at the beginning of each period

...



Part B: Future Value of Annuities

Future Value of Ordinary Annuity

Question: If you save 500 per month at 6% annual interest, how much will you have in 10 years?

...

! Future Value Formula (Ordinary Annuity)

$$FV = PMT \cdot \frac{(1+r)^n - 1}{r}$$

where:

- PMT = Payment per period
- r = Interest rate per period
- n = Total number of payments

FV Calculation Example

Given: 500/month, 6% annual (0.5% monthly), 10 years (120 months)

$$FV = 500 \cdot \frac{(1.005)^{120} - 1}{0.005}$$

...

$$FV = 500 \cdot \frac{1.8194 - 1}{0.005} = 500 \cdot \frac{0.8194}{0.005}$$

...

$$FV = 500 \cdot 163.88 = 81,939.67$$

...

> Tip
>
> Total deposits: $500 \times 120 = 60,000$
>
> Interest earned: $81,939.67 - 60,000 = 21,939.67$

FV of Annuity Due

Payments at beginning of period earn one extra period of interest:

...

> ! Future Value Formula (Annuity Due)
>
> $$FV_{\text{due}} = PMT \cdot \frac{(1+r)^n - 1}{r} \cdot (1+r)$$
>
> Or simply:
>
> $$FV_{\text{due}} = FV_{\text{ordinary}} \cdot (1+r)$$

...

Same example as annuity due:

$$FV = 81,939.67 \times 1.005 = 82,349.36$$

Part C: Present Value of Annuities

Present Value of Ordinary Annuity

Question: How much is a stream of future payments worth today?

...

> ! Present Value Formula (Ordinary Annuity)
>
> $$PV = PMT \cdot \frac{1-(1+r)^{-n}}{r}$$

...

Example: What is the present value of receiving 1,000/month for 5 years at 8% annual?

$$PV = 1000 \cdot \frac{1-(1.00667)^{-60}}{0.00667} = 1000 \cdot 49.32 = 49,318.43$$

PV of Annuity Due

> ! Present Value Formula (Annuity Due)
>
> $$PV_{\text{due}} = PMT \cdot \frac{1-(1+r)^{-n}}{r} \cdot (1+r)$$
>
> Or simply:
>
> $$PV_{\text{due}} = PV_{\text{ordinary}} \cdot (1+r)$$

...

Same example as annuity due:

$$PV = 49,318.43 \times 1.00667 = 49,647.32$$

Break - 10 Minutes

Part D: Solving for Unknowns

Finding the Payment Amount

Question: How much must you save monthly to have 100,000 in 15 years at 5%?

...

Rearrange the FV formula:

$$PMT = \frac{FV \cdot r}{(1+r)^n - 1}$$

...

$$PMT = \frac{100000 \times 0.004167}{(1.004167)^{180} - 1} = \frac{416.67}{1.1137} = 374.26$$

...

> ℹ Note
>
> You need to save 374.26 per month!

Finding Number of Payments

Question: How many months to pay off a 20,000 loan at 6% with 450/month payments?

. . .

From the PV formula, solve for n:

$$n = -\frac{\ln(1 - \frac{PV \cdot r}{PMT})}{\ln(1+r)}$$

. . .

$$n = -\frac{\ln(1 - \frac{20000 \times 0.005}{450})}{\ln(1.005)} = -\frac{\ln(0.7778)}{0.00499}$$

. . .

$$n = -\frac{-0.2513}{0.00499} = 50.4 \quad \text{months} \approx 51 \quad \text{payments}$$

Part E: Loan Amortization

What is Amortization?

Amortization = Paying off a loan through regular payments

. . .

Each payment consists of:

1. Interest portion: Pays interest on remaining balance
2. Principal portion: Reduces the loan balance

. . .

> ! Important
>
> Key insight: Early payments are mostly interest, later payments are mostly principal!

Loan Payment Formula

! Monthly Payment Formula

$$PMT = PV \cdot \frac{r}{1-(1+r)^{-n}}$$

where:

- PV = Loan amount (principal)
- r = Monthly interest rate
- n = Total number of payments

Payment Calculation Example

Loan: 200,000 at 4.5% annual for 30 years (mortgage)

$$r = 0.045/12 = 0.00375, \quad n = 360$$

...

$$PMT = 200000 \times \frac{0.00375}{1-(1.00375)^{-360}}$$

...

$$PMT = 200000 \times \frac{0.00375}{1-0.2598} = 200000 \times \frac{0.00375}{0.7402}$$

...

$$PMT = 200000 \times 0.005067 = 1,013.37$$

Building an Amortization Schedule I

First few payments of the 200,000 loan:

Payment	Payment	Interest	Principal	Balance
0	-	-	-	200,000.00
1	1,013.37	750.00	263.37	199,736.63
2	1,013.37	749.01	264.36	199,472.27
3	1,013.37	748.02	265.35	199,206.92
...	...	...	...	...

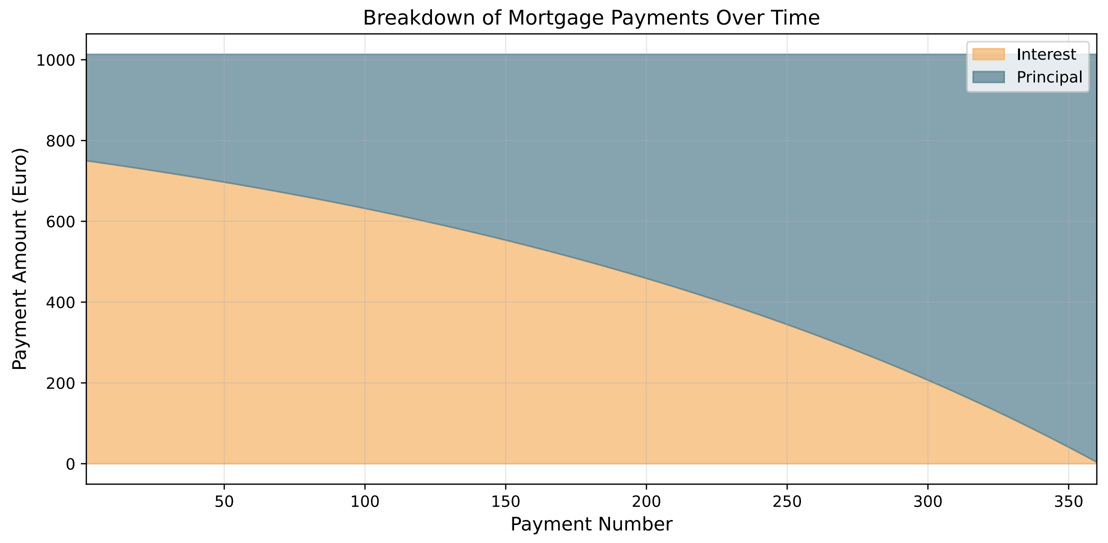
Building an Amortization Schedule II

How it works:

- Interest: Balance $\times$ rate = $200,000 \times 0.00375 = 750$
- Principal: Payment - Interest = $1,013.37 - 750 = 263.37$

- New Balance: Old Balance - Principal = $200,000 - 263.37$

Visualizing the Amortization



Outstanding Balance After k Payments

> ! Outstanding Balance Formula
>
> $$B_k = PV \cdot \frac{(1+r)^n - (1+r)^k}{(1+r)^n - 1}$$
>
> Or equivalently:
>
> $$B_k = PMT \cdot \frac{1 - (1+r)^{-(n-k)}}{r}$$

...

Example: Balance after 10 years (120 payments) on the 200,000 mortgage

$$B_{120} = 1013.37 \times \frac{1 - (1.00375)^{-240}}{0.00375} = 1013.37 \times 158.59 = 160,706.24$$

Part F: Applications

Retirement Planning

Scenario: You want 500,000 at retirement in 30 years. Investment earns 7% annually.

How much must you invest monthly?

...

$$PMT = \frac{FV \cdot r}{(1+r)^n - 1} = \frac{500000 \times 0.005833}{(1.005833)^{360} - 1}$$

. . .

$$PMT = \frac{2916.67}{7.878} = 370.16$$

. . .

> Tip
>
> Total invested: $370.16 \times 360 = 133,257.60$
>
> Interest earned: $500,000 - 133,257.60 = 366,742.40$ (more than 2.5x your contributions!)

Lease vs. Buy Decision

Should a company lease or buy equipment?

Buy: 50,000 now

Lease: 1,200/month for 4 years, then return

. . .

At 6% annual rate, PV of lease payments:

$$PV = 1200 \times \frac{1 - (1.005)^{-48}}{0.005} = 1200 \times 42.58 = 51,096.08$$

. . .

> Note
>
> Leasing costs 51,096 in present value terms - buying at 50,000 is cheaper (before considering resale value of owned equipment).

Guided Practice - 15 Minutes

Practice Problems

Work in pairs

Problem 1: Calculate the future value of saving 300/month for 20 years at 5% annual interest.

Problem 2: Find the monthly payment for a 25,000 car loan at 4.8% for 5 years.

Problem 3: Create the first 3 rows of an amortization schedule for a 10,000 loan at 6% annual for 2 years (monthly payments).

Problem 4: How much is a monthly pension of 2,000 for 25 years worth today at 4% annual interest?

Wrap-Up & Key Takeaways

Today's Essential Formulas

> **! Annuity Formulas**
>
> Future Value: $FV = PMT \cdot \frac{(1+r)^n - 1}{r}$
>
> Present Value: $PV = PMT \cdot \frac{1-(1+r)^{-n}}{r}$
>
> Payment: $PMT = PV \cdot \frac{r}{1-(1+r)^{-n}}$
>
> For annuity due: Multiply by $(1 + r)$

Key Takeaways

- Annuities are equal payments at regular intervals
- FV of annuity tells you what regular savings will become
- PV of annuity tells you what a payment stream is worth today
- Amortization shows how loans are paid off
- Early payments are mostly interest, later payments mostly principal

. . .

> **Coming Next**
>
> Session 08-03: Cost Analysis & Pricing Decisions - minimum pricing for profitability!

Homework Assignment

Tasks 08-02

- Calculate future and present values of annuities
- Solve for payment amounts and number of payments
- Build complete amortization schedules
- Apply concepts to retirement and loan decisions

. . .

> **Warning**
>
> These calculations require careful attention to the interest rate per period!