Session 08-01 - Compound Interest & Geometric Sequences

Section 08: Financial Mathematics

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Entry Quiz - 10 Minutes

Quick Review from Section 07

Test your understanding of Probability

1. In a survey, 60% of customers are satisfied, 40% are repeat buyers, and 30% are both. Find $P(\text{Satisfied OR Repeat})$.
2. A medical test has sensitivity 95% and specificity 90%. If disease prevalence is 2%, find the PPV.
3. A coin is flipped 4 times. Find $P(\text{exactly 2 heads})$.
4. Create a contingency table: 100 people surveyed, 55 own cars, 40 own bikes, 20 own both.

Welcome to Financial Mathematics!

New Section Overview

Section 08 covers essential financial topics:

- Session 08-01: Compound Interest & Geometric Sequences (today)
- Session 08-02: Annuities & Loan Amortization
- Session 08-03: Cost Analysis & Pricing Decisions

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> ! Important
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> Cost analysis and pricing decisions are exam-critical topics!

Learning Objectives

What You'll Master Today

- Understand geometric sequences and their formulas
- Calculate compound interest for different compounding periods

- Find effective annual rates for comparison
- Apply present value concepts to business decisions
- Use the Rule of 72 for quick estimates

Part A: Geometric Sequences

What is a Geometric Sequence?

A sequence where each term is multiplied by a constant ratio:

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$$a_1, a_1 \cdot r, a_1 \cdot r^2, a_1 \cdot r^3, \ldots$$

...

! Key Formulas

Explicit formula (nth term):

$$a_n = a_1 \cdot r^{n-1}$$

Recursive formula:

$$a_n = a_{n-1} \cdot r$$

Examples of Geometric Sequences

Common ratio determines behavior:

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$\boldsymbol{r = 2}$: $3, 6, 12, 24, 48, \ldots$ (growth)

$\boldsymbol{r = \frac{1}{2}}$: $16, 8, 4, 2, 1, \ldots$ (decay)

$\boldsymbol{r = -2}$: $1, -2, 4, -8, 16, \ldots$ (alternating)

...

Tip

- $|r| > 1$: sequence grows (or alternates with growth)
- $|r| < 1$: sequence decays toward zero
- $r = 1$: constant sequence

Part B: Compound Interest

Simple vs. Compound Interest

Two ways to grow money:

. . .

Simple Interest: Interest earned only on principal

$$A = P(1 + rt)$$

. . .

Compound Interest: Interest earned on principal AND accumulated interest

$$A = P(1 + r)^t$$

. . .

> ⚠ Warning
>
> Compound interest creates exponential growth - this is where geometric sequences appear!

The Compound Interest Formula

Annual Compounding

$$A = P(1 + r)^t$$

. . .

where:

- A = Final amount (future value)
- P = Principal (initial investment)
- r = Annual interest rate (as decimal)
- t = Time in years

. . .

Example: Invest 1,000 at 5% for 10 years, $A = 1000(1.05)^{10}$

Multiple Compounding Periods

What if interest compounds more frequently?

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General Compound Interest Formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where n = number of compounding periods per year

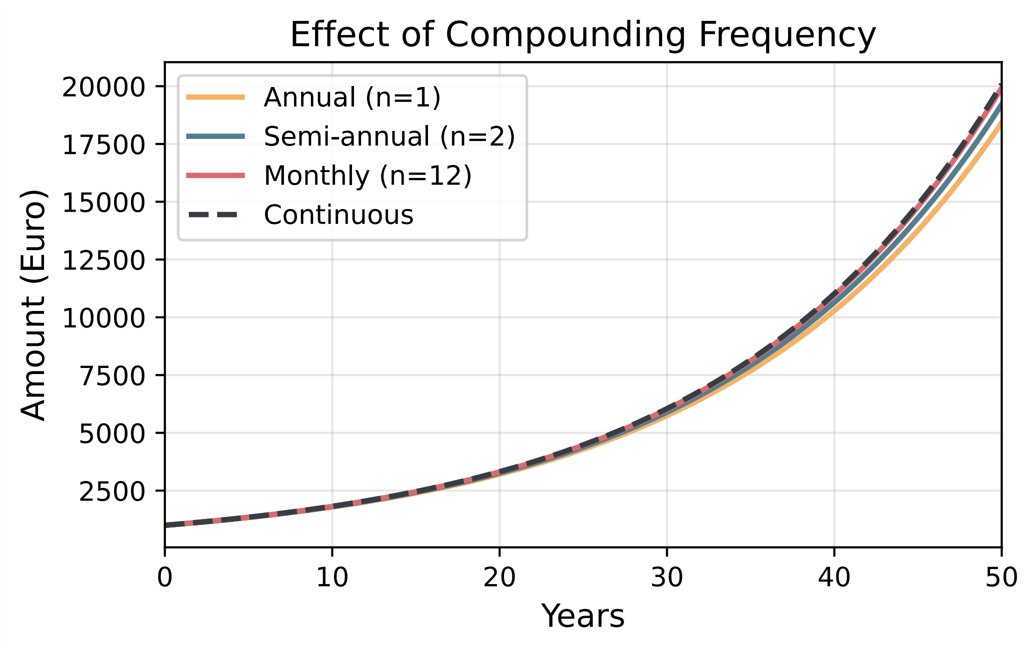
. . .

Tip

Compounding	n
Annual	1
Semi-annual	2
Quarterly	4
Monthly	12
Daily	365

Compounding Comparison

1,000 invested at 6% for 50 years:



Continuous Compounding

As $n \to \infty$, we get continuous compounding:

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Continuous Compounding Formula

$$A = Pe^{rt}$$

where $e \approx 2.71828$

...

Derivation: $\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^n = e^r$

. . .

Example: 1,000 at 6% continuous for 5 years, $A = 1000 \cdot e^{0.06 \times 5}$

Break - 10 Minutes

Part C: Effective Annual Rate

Comparing Different Rates

Problem: Bank A offers 6% compounded monthly. Bank B offers 6.1% compounded annually. Which is better?

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We need a common basis for comparison!

. . .

Effective Annual Rate (EAR)

$$r_{\text{eff}} = \left(1 + \frac{r}{n}\right)^n - 1$$

. . .

> Tip
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> This gives the equivalent annual rate for any compounding frequency.

EAR Calculation Example

Bank A: 6% compounded monthly

$$r_{\text{eff}} = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = (1.005)^{12} - 1 = 0.0617 = 6.17\%$$

. . .

Bank B: 6.1% compounded annually

$$r_{\text{eff}} = 6.1\%$$

. . .

> Tip
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> Bank A is slightly better (6.17% > 6.10%)!

EAR for Continuous Compounding

For continuous compounding:

. . .

$$r_{\text{eff}} = e^r - 1$$

. . .

Example: 6% continuous compounding

$$r_{\text{eff}} = e^{0.06} - 1 \approx 0.0618 = 6.18\%$$

Part D: Present Value

The Present Value Concept

Question: How much invest today to have 10,000 in 5 years at 6%?

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We need to discount future values back to today.

. . .

Present Value Formula

$$PV = \frac{FV}{(1+r)^t} = FV \cdot (1+r)^{-t}$$

Present Value Example

Goal: 10,000 in 5 years at 6%

$$PV = \frac{10000}{(1.06)^5} = \frac{10000}{1.3382} = 7,472.58$$

. . .

> **ℹ Note**
>
> You need to invest 7,472.58 today to have 10,000 in 5 years!

. . .

With monthly compounding:

$$PV = \frac{10000}{(1+0.06/12)^{60}} = \frac{10000}{1.3489} = 7,413.72$$

Part E: Business Applications

Investment Growth Analysis

A company invests 50,000 in bonds paying 4.5% compounded quarterly.

a) What is the value after 10 years?

b) What is the effective annual rate?
c) How long until the investment doubles?

Inflation Adjustment

Real Interest Rate (Fisher Equation)

$$r_{\text{real}} \approx r_{\text{nominal}} - r_{\text{inflation}}$$

. . .

Exact formula:

$$1 + r_{\text{real}} = \frac{1 + r_{\text{nominal}}}{1 + r_{\text{inflation}}}$$

. . .

Example: 6% nominal return, 2% inflation

$$r_{\text{real}} = \frac{1.06}{1.02} - 1 = 0.039 = 3.9\%$$

Guided Practice - 15 Minutes

Practice Problems

Work in pairs

Problem 1: Calculate the future value of 2,500 invested at 5.5% compounded monthly for 8 years.

Problem 2: Bank A offers 4.8% compounded daily. Bank B offers 4.9% compounded annually. Which bank offers a better return?

Problem 3: You need 25,000 in 6 years. How much must you invest today at 7% compounded quarterly?

Wrap-Up & Key Takeaways

Today's Essential Concepts

- Geometric sequences have constant ratio: $a_n = a_1 \cdot r^{n-1}$
- Compound interest creates exponential growth
- More frequent compounding increases returns
- Effective annual rate allows fair comparison
- Present value discounts future amounts
- Rule of 72 estimates doubling time quickly

. . .

Coming Next

Session 08-02: Annuities & Loan Amortization - regular payment streams!

Homework Assignment

Tasks 08-01

- Calculate geometric sequence terms and sums
- Compound interest with various compounding periods
- Compare investments using effective annual rates
- Present value calculations for financial planning

...

Note

Practice with your calculator - these calculations appear frequently on exams!