Session 07-07 - Binomial & Normal Distributions

Section 07: Probability & Statistics

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Entry Quiz - 10 Minutes

Quick Review from Session 07-06

1. From a contingency table, how do you calculate $P(A \mid B)$?
2. In a table with 200 total, 80 in category A, 60 in category B, and 30 in both. Find $P(A \mid B)$.
3. How do you test if two variables are independent using a table?
4. A company has 1000 employees: 600 full-time, 400 with degrees, 280 full-time with degrees. Build the table.

Learning Objectives

What You'll Master Today

- Identify binomial experiments and their requirements
- Apply the binomial formula: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
- Calculate probabilities: "exactly k", "at most k", "at least k"
- Use the geometric distribution for "first success" problems
- Understand the normal distribution basics

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> **!** Important
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> Binomial distribution problems appear on every Feststellungsprüfung!

Part A: Binomial Experiments

Requirements for Binomial

! Binomial Experiment Conditions

1. Fixed number of trials: n is known in advance
2. Two outcomes: Success (probability p) or Failure (probability $1 - p$)
3. Independence: Trials don't affect each other
4. Constant probability: p is the same for all trials

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Examples: - Flipping a coin 10 times (heads = success) - Testing 50 products (defective = success) - Surveying 100 customers (satisfied = success)

The Binomial Formula

! Binomial Distribution

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Where: - n = number of trials - k = number of successes - p = probability of success - $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ = number of ways

Understanding the Formula

$$P(X = k) = \underbrace{\binom{n}{k}}_{\text{arrangements}} \times \underbrace{p^k}_{\text{k successes}} \times \underbrace{(1-p)^{n-k}}_{\text{n-k failures}}$$

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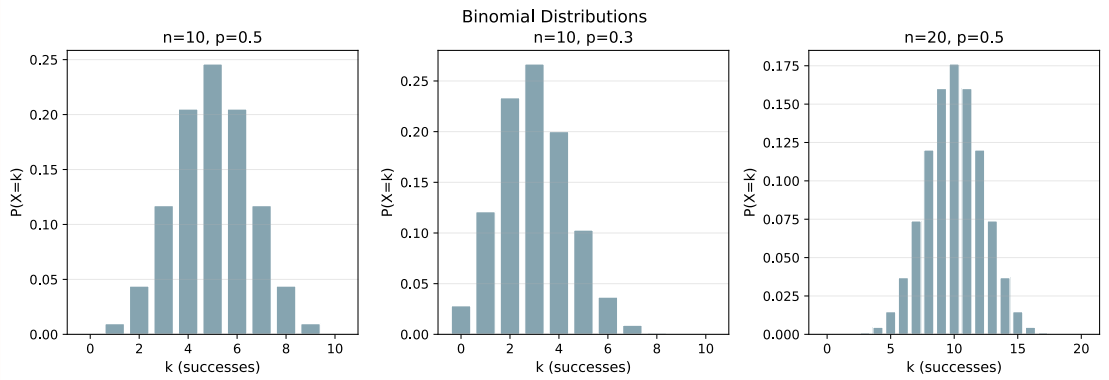
Example: In 5 coin flips, $P(\text{exactly } 3 \text{ heads})$?

$$P(X = 3) = \binom{5}{3} \times (0.5)^3 \times (0.5)^2 = 10 \times 0.125 \times 0.25 = 0.3125$$

Binomial Distribution Visualization



Part B: Common Probability Questions

Three Types of Questions

Question Type	Formula
Exactly k	$P(X = k)$
At most k	$P(X \leq k) = \sum_{i=0}^{k} P(X = i)$
At least k	$P(X \geq k) = 1 - P(X < k) = 1 - P(X \leq k-1)$

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> **Tip**
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> For "at least" problems, use the complement rule!

Example: Quality Control

A machine produces items with 8% defect rate. In a batch of 15 items:

a) $P(\text{exactly 2 defective})$

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$$P(X = 2) = \binom{15}{2}(0.08)^2(0.92)^{13} = 105 \times 0.0064 \times 0.326 \approx 0.219$$

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b) $P(\text{at most 1 defective})$

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$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= \binom{15}{0}(0.08)^0(0.92)^{15} + \binom{15}{1}(0.08)^1(0.92)^{14}$$

$$= 0.286 + 0.373 = 0.659$$

Example Continued

c) $P(\text{at least 2 defective})$

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$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - 0.659 = 0.341$$

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d) $P(\text{between 1 and 3 defective, inclusive})$

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$$P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$\approx 0.373 + 0.219 + 0.085 = 0.677$$

Expected Value and Variance

! Binomial Mean and Variance

- Expected value (mean): $\mu = E[X] = np$
- Variance: $\sigma^2 = np(1-p)$
- Standard deviation: $\sigma = \sqrt{np(1-p)}$

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Example: If $n = 100$ and $p = 0.3$: - Expected successes: $\mu = 100 \times 0.3 = 30$ - Standard deviation: $\sigma = \sqrt{100 \times 0.3 \times 0.7} = \sqrt{21} \approx 4.58$

Break - 10 Minutes

Part C: Geometric Distribution

Waiting for First Success

! Geometric Distribution

The probability that the first success occurs on trial n:

$$P(X = n) = (1-p)^{n-1} \cdot p$$

Where p = probability of success on each trial.

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Example: A salesperson has a 20% chance of making a sale on each call. What's the probability the first sale is on the 4th call?

$$P(X = 4) = (0.8)^3 \times 0.2 = 0.512 \times 0.2 = 0.1024$$

Geometric: Expected Trials

Expected Number of Trials

For the geometric distribution:

$$E[X] = \frac{1}{p}$$

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Example: If success probability is 0.25, on average how many trials until first success?

$$E[X] = \frac{1}{0.25} = 4 \text{ trials}$$

Geometric Example: Exam Style

A machine produces defective items with probability 0.05.

a) $P(\text{first defective item is the 10th produced})$

$$P(X = 10) = (0.95)^9 \times 0.05 = 0.631 \times 0.05 = 0.0316$$

. . .

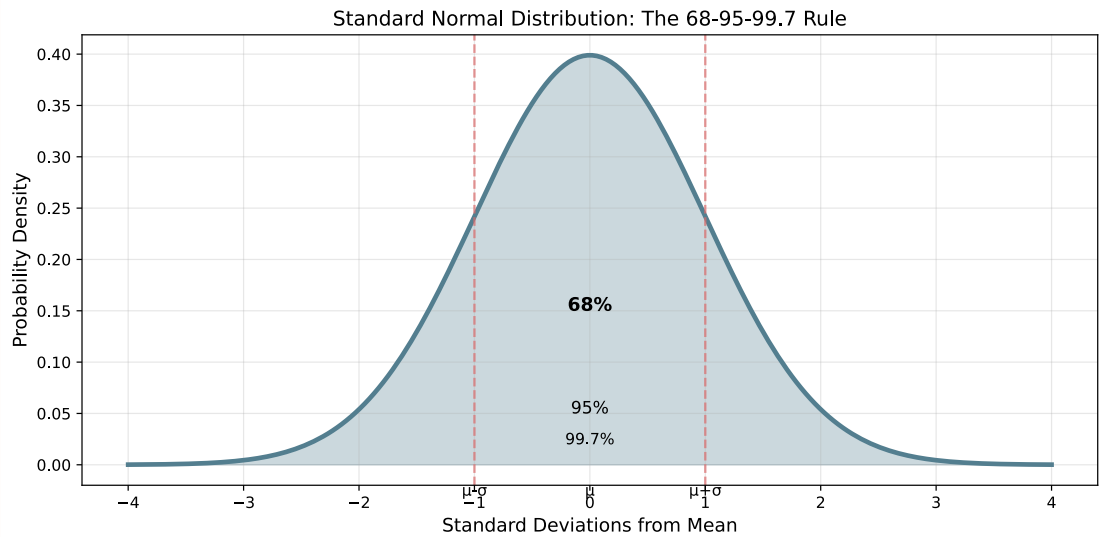
b) $P(\text{first defective item within first 5 items})$

$$P(X \leq 5) = 1 - P(\text{no defective in first 5}) = 1 - (0.95)^5$$

$$= 1 - 0.774 = 0.226$$

Part D: Normal Distribution Basics

The Bell Curve



The 68-95-99.7 Rule

> **!** Empirical Rule
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> For normal distributions: - 68% of data falls within $\mu \pm 1\sigma$ - 95% of data falls within $\mu \pm 2\sigma$ - 99.7% of data falls within $\mu \pm 3\sigma$

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Example: Test scores have $\mu = 75$ and $\sigma = 10$

- 68% of students score between 65 and 85
- 95% of students score between 55 and 95
- 99.7% of students score between 45 and 105

Normal Approximation to Binomial

When n is large and p is not too extreme:

$$\text{Binomial } (n, p) \approx \text{Normal } \left(\mu = np, \sigma = \sqrt{np(1-p)}\right)$$

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Rule of thumb: Use when $np \geq 5$ and $n(1-p) \geq 5$

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Example: If $n = 100$ and $p = 0.4$: - $\mu = 40$ - $\sigma = \sqrt{100 \times 0.4 \times 0.6} = \sqrt{24} \approx 4.9$ - 95% of samples will have between $40 - 9.8 = 30.2$ and $40 + 9.8 = 49.8$ successes

Guided Practice - 20 Minutes

Practice Problem 1

A multiple choice test has 20 questions with 4 options each. A student guesses randomly on all questions.

a) What's the probability of getting exactly 5 correct?
b) What's the probability of getting at least 8 correct?
c) What's the expected number of correct answers?
d) What's the standard deviation?

Practice Problem 2 (Exam Style)

A company's call center receives calls with 15% conversion rate.

a) In 10 calls, what's the probability of exactly 2 conversions?
b) In 10 calls, what's the probability of at least 1 conversion?
c) What's the probability that the first conversion is on the 5th call?
d) On average, how many calls until the first conversion?

Wrap-Up & Key Takeaways

Today's Essential Concepts

- Binomial formula: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
- Binomial parameters: $\mu = np$, $\sigma = \sqrt{np(1-p)}$
- Three question types: Exactly k, at most k, at least k
- Geometric: $(1-p)^{n-1} \cdot p$ for first success on trial n
- Normal: 68-95-99.7 rule for bell curves

Next Session Preview

Coming Up: Mock Exam 2

- Full 180-minute exam simulation
- All Section 07 topics covered
- Practice under exam conditions
- Prepare your formula sheet!

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Homework

Complete Tasks 07-07 - focus on binomial calculation practice!