Session 07-05 - Bayes' Theorem

Section 07: Probability & Statistics

Dr. Nikolai Heinrichs & Dr. Tobias Vlćek

Entry Quiz - 10 Minutes

Quick Review from Session 07-04

1. If $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \cap B) = 0.2$, find $P(A \mid B)$.
2. A bag has 4 red and 6 blue balls. Two are drawn without replacement. Find $P(\text{both blue})$.
3. Given $P(B \mid A) = 0.6$ and $P(A) = 0.3$, find $P(A \cap B)$.
4. If $P(A \mid B) = P(A)$, what can we conclude about A and B?

Learning Objectives

What You'll Master Today

- Apply Bayes' Theorem: $P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$
- Understand prior and posterior probabilities
- Calculate sensitivity and specificity for diagnostic tests
- Compute PPV and NPV (positive/negative predictive values)
- Solve medical testing problems - a key exam topic!

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> **! Important**
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> Bayes' Theorem appears on virtually every Feststellungsprüfung!

Part A: Bayes' Theorem

Reversing Conditional Probabilities

The problem: We often know $P(B \mid A)$ but need $P(A \mid B)$.

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Example: - We know $P(\text{positive test} \mid \text{disease})$ (sensitivity) - We need $P(\text{disease} \mid \text{positive test})$ (PPV)

. . .

i Note

These are not the same! This is a common misconception.

Bayes' Theorem Formula

! Bayes' Theorem (Satz von Bayes)

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

Using the law of total probability:

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B \mid A) \cdot P(A) + P(B \mid A') \cdot P(A')}$$

Understanding the Components

Term	Name	Meaning
$P(A)$	Prior	Initial probability before evidence
$P(A \| B)$	Posterior	Updated probability after evidence
$P(B \| A)$	Likelihood	How likely is evidence given A?
$P(B)$	Evidence	Total probability of evidence

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$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}$$

Part B: Medical Testing Framework

Key Terminology

! Medical Test Metrics

Metric	Formula	Meaning
Sensitivity	$P(+ \| D)$	Correctly identifies sick people
Specificity	$P(- \| D')$	Correctly identifies healthy people
Prevalence	$P(D)$	Proportion with disease in population
PPV	$P(D \| +)$	Probability of disease given positive test
NPV	$P(D' \| -)$	Probability of no disease given negative test

The 2×2 Table

	Disease (+)	No Disease (–)	Total
Test +	True Positive (TP)	False Positive (FP)	Test +
Test –	False Negative (FN)	True Negative (TN)	Test –
Total	Disease	No Disease	Population

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- Sensitivity = $\frac{TP}{TP+FN}$
- Specificity = $\frac{TN}{TN+FP}$
- PPV = $\frac{TP}{TP+FP}$
- NPV = $\frac{TN}{TN+FN}$

Example: COVID Test

A rapid COVID test has:

- Sensitivity: 95% (correctly identifies 95% of infected people)
- Specificity: 98% (correctly identifies 98% of healthy people)
- Prevalence: 2% (2% of population currently infected)

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Question: If you test positive, what's the probability you actually have COVID?

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This is asking for PPV = $P(D \mid +)$!

Solution Using Bayes' Theorem

$$P(D \mid +) = \frac{P(+ \mid D) \cdot P(D)}{P(+)}$$

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Calculate $\boldsymbol{P(+)}$ using law of total probability:

$$P(+) = P(+ \mid D) \cdot P(D) + P(+ \mid D') \cdot P(D')$$

$$= 0.95 \times 0.02 + 0.02 \times 0.98 = 0.019 + 0.0196 = 0.0386$$

. . .

Apply Bayes:

$$P(D \mid +) = \frac{0.95 \times 0.02}{0.0386} = \frac{0.019}{0.0386} \approx 0.492$$

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> ⚠ Warning
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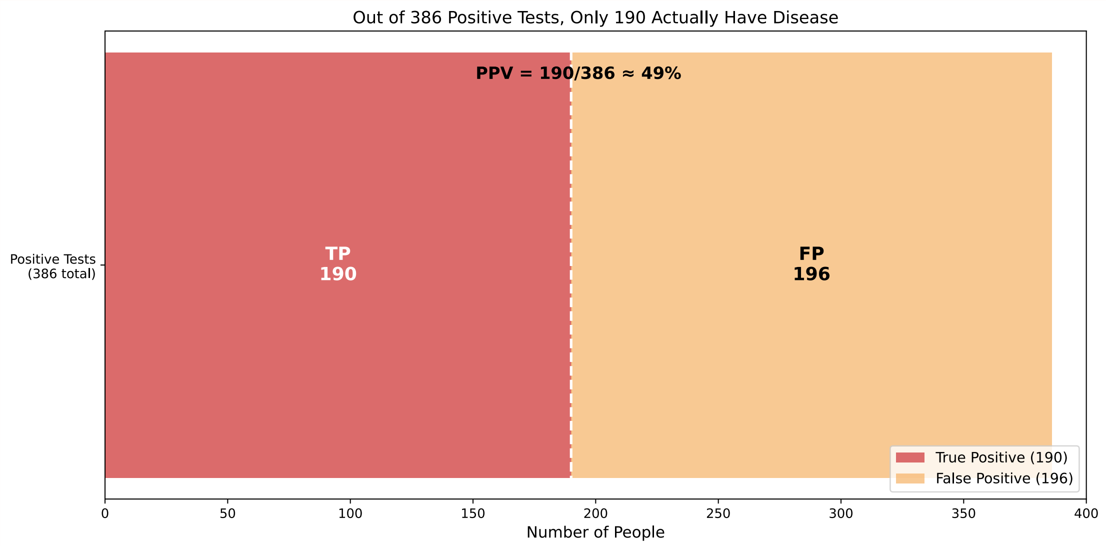
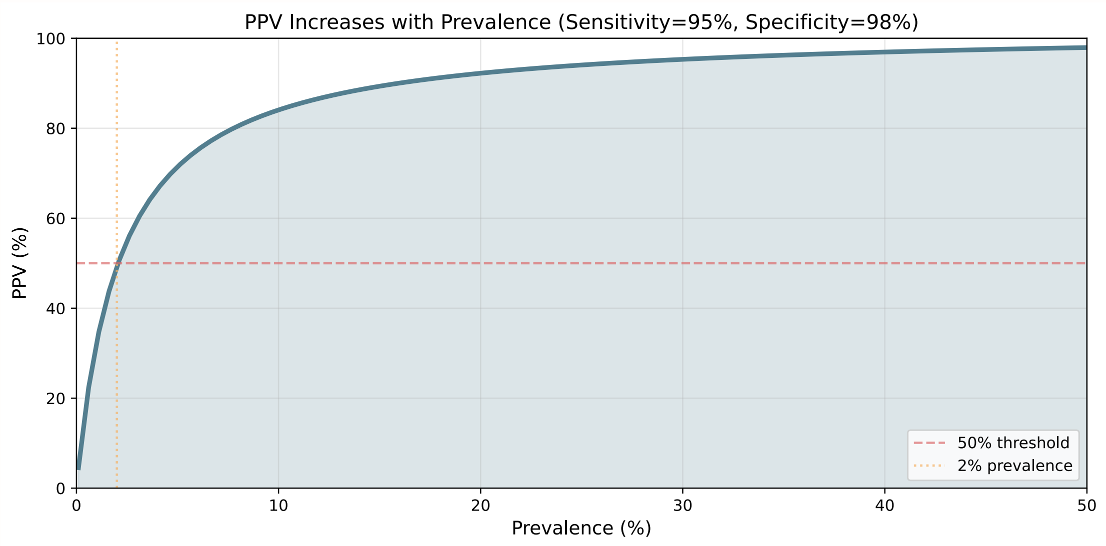
> Only about 49% of positive tests are true positives when prevalence is low!

Visual: Why PPV Can Be Low



The Prevalence Effect



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PPV depends heavily on prevalence!

Break - 10 Minutes

Part C: Systematic Problem-Solving

Step-by-Step Approach

Strategy for Bayes Problems

1. Identify what you need: Usually $P(D \mid +)$ or $P(D \mid -)$
2. Extract given information: Sensitivity, specificity, prevalence
3. Set up the formula: Write Bayes' theorem
4. Calculate $P(+)$ or $P(-)$: Use law of total probability
5. Substitute and solve: Careful with arithmetic!
6. Interpret: What does the answer mean?

Complete Example: Disease Screening

A screening test for a disease has:

- Sensitivity = 90%
- Specificity = 95%
- Prevalence = 1%

Find: a) PPV b) NPV

Solution Part a) PPV

$$P(D \mid +) = \frac{P(+ \mid D) \cdot P(D)}{P(+ \mid D) \cdot P(D) + P(+ \mid D') \cdot P(D')}$$

...

Given values: - $P(+ \mid D) = 0.90$ (sensitivity) - $P(D) = 0.01$ (prevalence) - $P(+ \mid D') = 1 - 0.95 = 0.05$ (false positive rate) - $P(D') = 0.99$

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$$P(D \mid +) = \frac{0.90 \times 0.01}{0.90 \times 0.01 + 0.05 \times 0.99}$$

$$= \frac{0.009}{0.009 + 0.0495} = \frac{0.009}{0.0585} \approx 0.154$$

Solution Part b) NPV

$$P(D' \mid -) = \frac{P(- \mid D') \cdot P(D')}{P(-)}$$

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Calculate $P(-)$:

$$P(-) = P(- \mid D) \cdot P(D) + P(- \mid D') \cdot P(D')$$

$$= 0.10 \times 0.01 + 0.95 \times 0.99 = 0.001 + 0.9405 = 0.9415$$

. . .

$$P(D' \mid -) = \frac{0.95 \times 0.99}{0.9415} = \frac{0.9405}{0.9415} \approx 0.999$$

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i Note

PPV is only 15.4%, but NPV is 99.9%! A negative result is very reliable.

Part D: Contingency Table Method

Alternative Approach

Use a hypothetical population (e.g., 10,000 people):

	Disease	No Disease	Total
Test +			
Test –			
Total	100	9,900	10,000

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Fill in using sensitivity and specificity:

	Disease	No Disease	Total
Test +	90	495	585
Test –	10	9,405	9,415
Total	100	9,900	10,000

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Read directly: PPV = $\frac{90}{585} = 0.154$, NPV = $\frac{9405}{9415} = 0.999$

Guided Practice - 20 Minutes

Practice Problem 1

A factory has two machines:

- Machine A produces 60% of items, with 3% defect rate
- Machine B produces 40% of items, with 5% defect rate

If a randomly selected item is defective, what's the probability it came from Machine A?

Practice Problem 2 (Exam-Style)

A medical test has sensitivity 85% and specificity 90%.

In a population with 5% prevalence:

a) Calculate PPV
b) Calculate NPV
c) Construct a contingency table for 1000 people
d) Interpret your results

Wrap-Up & Key Takeaways

Today's Essential Concepts

- Bayes' Theorem: $P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$
- Medical testing: Sensitivity, specificity, prevalence
- PPV and NPV: What positive/negative results mean
- Prevalence matters: Low prevalence $\longrightarrow$ Low PPV
- Two methods: Formula or contingency table

Next Session Preview

Coming Up: Contingency Tables

- Constructing tables from word problems
- Reading marginal, joint, and conditional probabilities
- Independence testing in tables
- Exam-style problems

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> Homework
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> Complete Tasks 07-05 - especially the medical testing problems!