Session 07-04 - Conditional Probability

Section 07: Probability & Statistics

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Entry Quiz - 10 Minutes

Quick Review from Session 07-03

Test your understanding of Combinatorics

1. Calculate $\binom{8}{3}$
2. How many ways can 5 people line up in a row?
3. A committee of 3 is chosen from 10 people. How many ways?
4. What's the probability of getting exactly 3 heads in 5 coin flips? (Use combinations)

Learning Objectives

What You'll Master Today

- Define conditional probability: $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
- Apply the multiplication rule: $P(A \cap B) = P(A \mid B) \cdot P(B)$
- Construct and use tree diagrams for sequential events
- Apply the law of total probability
- Test for independence using conditional probability

. . .

> **! Important**
>
> Conditional probability is heavily tested on the Feststellungsprüfung!

Part A: Conditional Probability Concept

Motivation

Question: A card is drawn from a deck. What's the probability it's a king?

. . .

$$P(\text{King}) = \frac{4}{52} = \frac{1}{13}$$

. . .

Question: Given that the card is a face card, what's the probability it's a king?

. . .

The condition changes the sample space!

$$P(\text{King} \mid \text{Face card}) = \frac{4}{12} = \frac{1}{3}$$

Conditional Probability Definition

! Definition: Conditional Probability

The probability of A given that B has occurred:

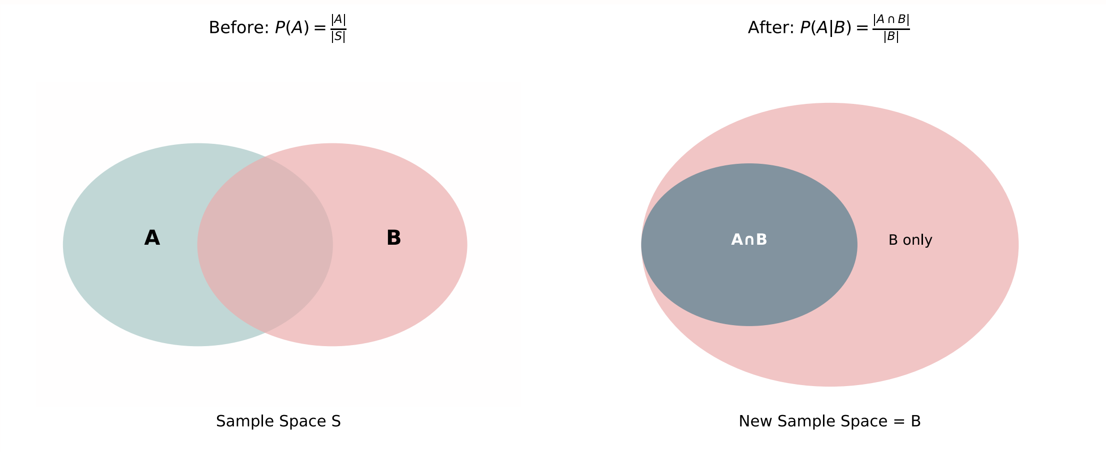
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

where $P(B) > 0$.

. . .

Read "$P(A \mid B)$" as "probability of A given B"

Visual Interpretation



Example Calculation

In a company: 60% work full-time, 30% work full-time AND have a degree.

Question: What's the probability an employee has a degree, given they work full-time?

. . .

Let D = has degree, F = full-time

$$P(D \mid F) = \frac{P(D \cap F)}{P(F)} = \frac{0.30}{0.60} = 0.50$$

. . .

> **i Note**
>
> Given that someone works full-time, there's a 50% chance they have a degree.

Part B: Multiplication Rule

From Conditional to Joint Probability

Rearranging the definition:

> **! Multiplication Rule**
>
> $$P(A \cap B) = P(A \mid B) \cdot P(B)$$
>
> or equivalently:
>
> $$P(A \cap B) = P(B \mid A) \cdot P(A)$$

. . .

Both formulas give the same result!

Example: Sequential Selection

A box contains 5 red and 3 blue balls. Two balls are drawn without replacement.

Question: What's the probability both are red?

. . .

Let R_1 = first ball red, R_2 = second ball red

$$P(R_1 \cap R_2) = P(R_2 \mid R_1) \cdot P(R_1)$$

. . .

$$= \frac{4}{7} \times \frac{5}{8} = \frac{20}{56} = \frac{5}{14}$$

Extended Multiplication Rule

For three or more events:

$$P(A \cap B \cap C) = P(A) \cdot P(B \mid A) \cdot P(C \mid A \cap B)$$

. . .

Example: Drawing 3 cards without replacement

$$P(3 \text{ aces}) = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} = \frac{24}{132,600} = \frac{1}{5,525}$$

Part C: Tree Diagrams

Visual Tool for Sequential Events

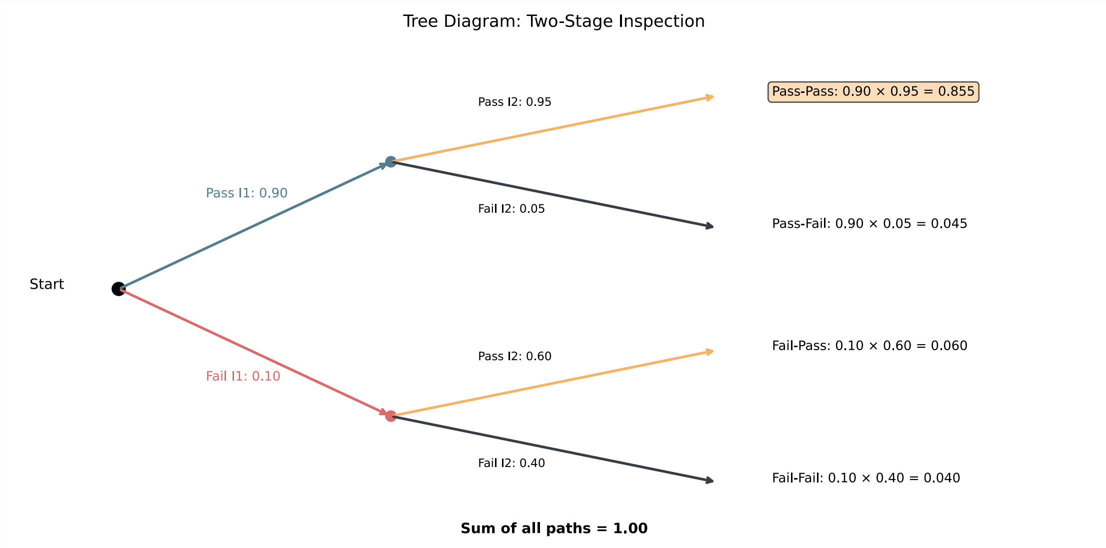
Tree diagrams show:

- Branches = outcomes at each stage
- Branch labels = probabilities (conditional if sequential)
- Multiply along paths = joint probability
- Add across paths = union probability

Example: Quality Control

A machine produces items that pass through 2 inspections.

- 90% pass inspection 1
- Of those passing inspection 1: 95% pass inspection 2
- Of those failing inspection 1: 60% pass inspection 2



Reading Tree Diagrams

From the previous example:

. . .

P(item passes both inspections): Follow Pass-Pass path

$$0.90 \times 0.95 = 0.855$$

. . .

P(item passes inspection 2): Add all paths ending with Pass I2

$$0.855 + 0.060 = 0.915$$

. . .

P(passed I1 | passed I2): (We'll use Bayes for this later!)

$$P(I_1 \mid I_2) = \frac{P(I_1 \cap I_2)}{P(I_2)} = \frac{0.855}{0.915} \approx 0.934$$

Break - 10 Minutes

Part D: Law of Total Probability

Partitioning the Sample Space

> ! Law of Total Probability
>
> If events $B_1, B_2, ..., B_n$ partition the sample space (mutually exclusive and exhaustive):
>
> $$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$$

. . .

Common case with two partitions:

$$P(A) = P(A \mid B) \cdot P(B) + P(A \mid B') \cdot P(B')$$

Example: Market Segments

A company sells to three market segments:

- Segment A: 50% of customers, 20% buy premium
- Segment B: 30% of customers, 35% buy premium
- Segment C: 20% of customers, 50% buy premium

. . .

Question: What percentage of all customers buy premium?

. . .

$$P(\text{Premium}) = 0.20(0.50) + 0.35(0.30) + 0.50(0.20)$$

$$= 0.10 + 0.105 + 0.10 = 0.305$$

30.5% of customers buy premium products.

Part E: Independence Revisited

Testing Independence

! Independence Test

Events A and B are independent if and only if:

$$P(A \mid B) = P(A)$$

(Learning B occurred doesn't change the probability of A)

. . .

Equivalently: $P(B \mid A) = P(B)$ or $P(A \cap B) = P(A) \cdot P(B)$

Example: Testing Independence

Survey data shows:

- $P(\text{Exercise regularly}) = 0.40$
- $P(\text{Healthy weight}) = 0.55$
- $P(\text{Healthy weight} \mid \text{Exercise}) = 0.70$

. . .

Question: Are "exercise regularly" and "healthy weight" independent?

. . .

Test: Is $P(\text{Healthy} \mid \text{Exercise}) = P(\text{Healthy})$?

$0.70 \neq 0.55$

. . .

Conclusion: Events are NOT independent - exercise is associated with healthy weight.

Guided Practice - 20 Minutes

Practice Problems

Work in pairs

Problem 1: 70% of students study math, 60% study economics, and 45% study both. a) Find $P(\text{Econ} \mid \text{Math})$ b) Find $P(\text{Math} \mid \text{Econ})$ c) Are these events independent?

Problem 2: Draw a tree diagram for: A bag has 3 red and 2 blue marbles. Two marbles are drawn without replacement. Find $P(\text{both same color})$.

Wrap-Up & Key Takeaways

Today's Essential Concepts

- Conditional probability: $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
- Multiplication rule: $P(A \cap B) = P(A \mid B) \cdot P(B)$
- Tree diagrams: Multiply along paths, add across paths
- Total probability: Sum over all branches
- Independence: $P(A \mid B) = P(A)$

Next Session Preview

Coming Up: Bayes' Theorem

- Reversing conditional probabilities
- Prior and posterior probabilities
- Medical testing applications (sensitivity, specificity)
- Business decision making

. . .

> **Homework**
>
> Complete Tasks 07-04 - focus on tree diagrams and conditional probability!