Session 07-03 - Combinatorics & Counting

Section 07: Probability & Statistics

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Entry Quiz - 10 Minutes

Quick Review from Session 07-02

Test your understanding of Basic Probability

1. If $P(A) = 0.4$ and $P(B) = 0.3$ and A and B are independent, find $P(A \cap B)$.
2. Use the complement rule: If $P(\text{rain}) = 0.25$, find $P(\text{no rain})$.
3. If $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \cap B) = 0.2$, find $P(A \cup B)$.
4. Are events A and B from question 3 independent? Why or why not?

Learning Objectives

What You'll Master Today

- Apply the fundamental counting principle for sequential events
- Calculate permutations: $P(n,r) = \frac{n!}{(n-r)!}$
- Calculate combinations: $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Distinguish when order matters vs. doesn't matter
- Connect counting to probability calculations

. . .

> ! Important
>
> Combinatorics is essential for calculating probabilities on the exam!

Part A: Fundamental Counting Principle

Sequential Choices

If you make sequential choices with $n_1, n_2, ..., n_k$ options:

$$\text{Total possibilities} = n_1 \times n_2 \times ... \times n_k$$

. . .

Example: Creating an outfit

- 4 shirts
- 3 pairs of pants
- 2 pairs of shoes

. . .

Total outfits: $4 \times 3 \times 2 = 24$

License Plate Example

A license plate has:

- 3 letters (A-Z, 26 options each)
- 4 digits (0-9, 10 options each)

. . .

Question: How many different plates are possible?

. . .

With repetition allowed:

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^3 \times 10^4 = 175,760,000$$

. . .

Without repetition:

$$26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$$

Factorial Notation

! Definition: Factorial

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

Special cases: $0! = 1$ and $1! = 1$

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Examples: - $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ - $3! = 3 \times 2 \times 1 = 6$ - $10! = 3,628,800$

Part B: Permutations

When Order Matters

> ! Definition: Permutation
>
> A permutation is an arrangement of objects where order matters.
>
> $$P(n, r) = \frac{n!}{(n-r)!}$$
>
> = Number of ways to arrange r objects from n distinct objects

...

Example: How many ways can 3 people win gold, silver, and bronze from 8 competitors?

$$P(8,3) = \frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5!}{5!} = 8 \times 7 \times 6 = 336$$

Permutation Examples

Example 1: Arrange all letters in "MATH"

...

$$P(4,4) = 4! = 24 \quad \text{arrangements}$$

...

Example 2: How many 4-digit PINs with no repeated digits?

...

$$P(10,4) = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7 = 5,040$$

...

Example 3: A president, VP, and treasurer from 12 candidates?

...

$$P(12,3) = 12 \times 11 \times 10 = 1,320$$

Part C: Combinations

When Order Doesn't Matter

! Definition: Combination

A combination is a selection of objects where order doesn't matter.

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

= Number of ways to choose r objects from n distinct objects

. . .

Key insight: Each combination corresponds to $r!$ permutations

$$C(n,r) = \frac{P(n,r)}{r!}$$

Combination Examples

Example 1: Choose 3 students from 10 for a committee

. . .

$$C(10,3) = \binom{10}{3} = \frac{10!}{3! \cdot 7!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$$

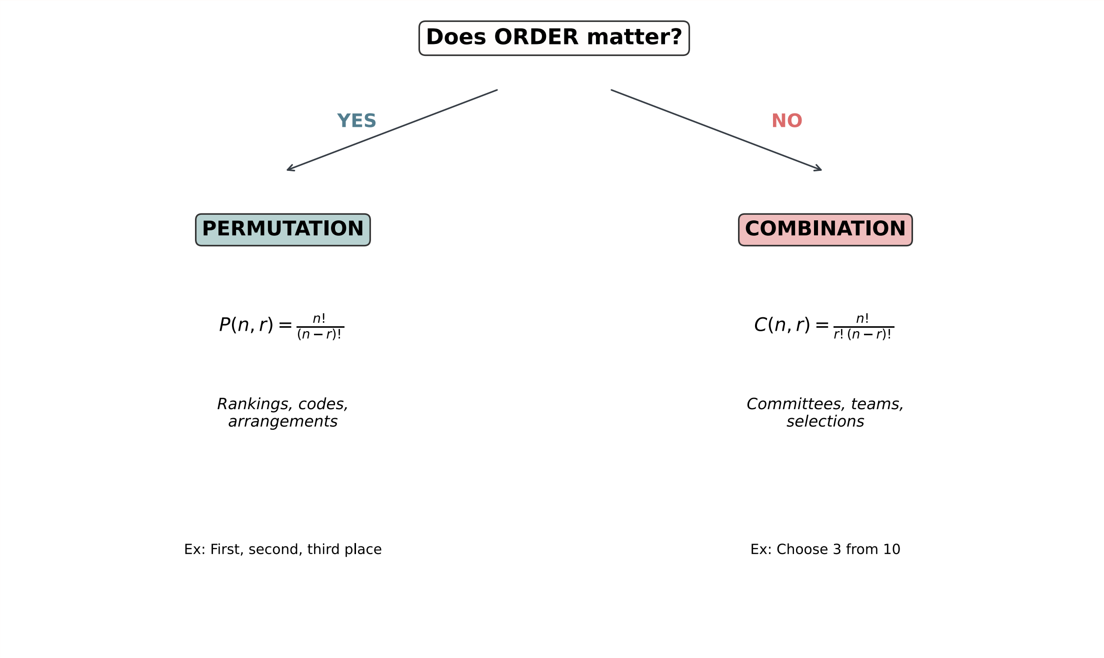
. . .

Example 2: Choose 5 cards from a 52-card deck

. . .

$$C(52,5) = \binom{52}{5} = \frac{52!}{5! \cdot 47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{120} = 2,598,960$$

Permutation vs Combination Decision Tree



Break - 10 Minutes

Part D: Combinatorics in Probability

Connecting Counting to Probability

For equally likely outcomes:

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

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Counting helps us find both numbers!

Example: Card Probability

What is the probability of being dealt exactly 2 hearts in a 5-card hand?

. . .

Favorable outcomes: Choose 2 hearts from 13, AND 3 non-hearts from 39

$$\binom{13}{2} \times \binom{39}{3} = 78 \times 9,139 = 712,842$$

. . .

Total outcomes: Choose any 5 cards from 52

$$\binom{52}{5} = 2,598,960$$

...

Probability:

$$P(\text{exactly 2 hearts}) = \frac{712,842}{2,598,960} \approx 0.274$$

Example: Committee Selection

A committee of 4 must be chosen from 6 men and 5 women.

Question: What is the probability that the committee has exactly 2 women?

...

Total ways to form committee: $\binom{11}{4} = 330$

...

Ways to get exactly 2 women: $\binom{5}{2} \times \binom{6}{2} = 10 \times 15 = 150$

...

Probability:

$$P(\text{exactly 2 women}) = \frac{150}{330} = \frac{5}{11} \approx 0.455$$

Example: Quality Control

A box contains 20 items: 4 are defective. If we select 3 items randomly:

a) $P(\text{none defective})$

...

$$P = \frac{\binom{16}{3}}{\binom{20}{3}} = \frac{560}{1140} = \frac{14}{28.5} \approx 0.491$$

...

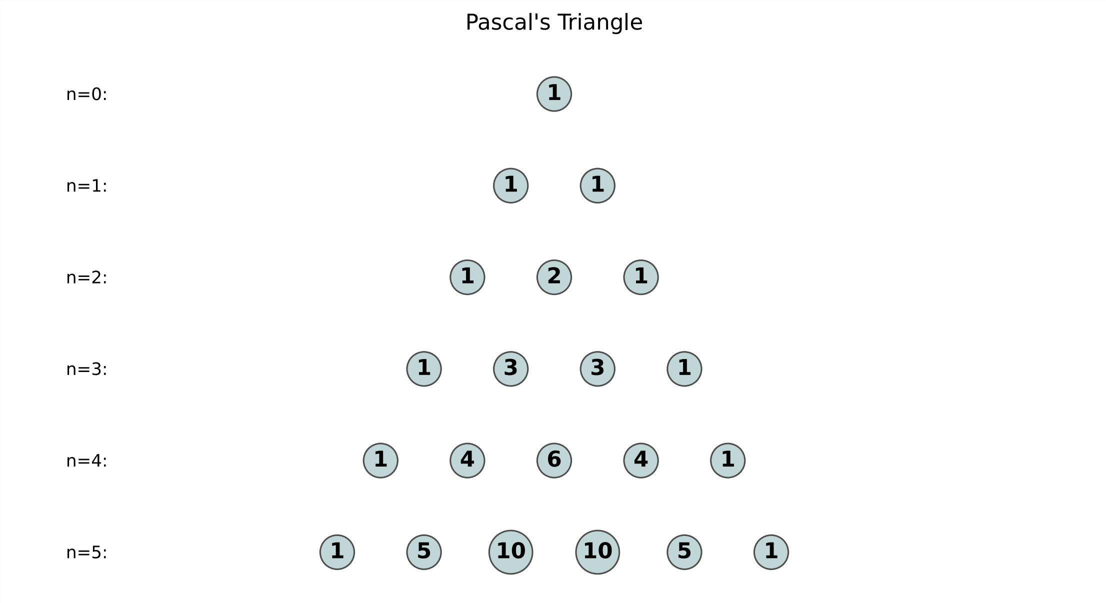
b) $P(\text{at least one defective})$

...

$$P = 1 - P(\text{none}) = 1 - 0.491 = 0.509$$

Part E: Pascal's Triangle

Binomial Coefficients Pattern



Each number is $\binom{n}{r}$ and equals the sum of the two numbers above it

...

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Guided Practice - 20 Minutes

Practice Problems

Work in pairs

Problem 1: A password must have 3 letters followed by 2 digits. a) How many passwords are possible (with repetition)? b) How many if no repetition is allowed?

Problem 2: From 8 candidates, we select a president, VP, secretary, and treasurer. How many ways?

Problem 3: A lottery requires choosing 6 numbers from 49. How many possible combinations?

Wrap-Up & Key Takeaways

Today's Essential Concepts

- Counting principle: Multiply choices for sequential events
- Factorial: $n! = n \times (n-1) \times \ldots \times 1$
- Permutations: Order matters - $P(n,r) = \frac{n!}{(n-r)!}$

- Combinations: Order doesn't matter - $C(n,r) = \frac{n!}{r!(n-r)!}$
- Key question: Does the order of selection matter?

Next Session Preview

Coming Up: Conditional Probability

- Conditional probability: $P(A \mid B)$
- Multiplication rule
- Tree diagrams
- Independence revisited

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Homework

Complete Tasks 07-03: - Practice permutation and combination calculations - Solve probability problems using combinatorics