Session 07-02 - Basic Probability Concepts

Section 07: Probability & Statistics

Dr. Nikolai Heinrichs & Dr. Tobias Vlćek

Entry Quiz - 10 Minutes

Quick Review from Session 07-01

Test your understanding of Descriptive Statistics

1. Find the mean and median of: $8, 12, 15, 9, 16, 12, 11$
2. If the variance of a dataset is 16, what is the standard deviation?
3. A frequency table shows 15 out of 50 items are defective. What is the relative frequency of defective items?
4. What is the interquartile range if $Q1 = 25$ and $Q3 = 45$?

Learning Objectives

What You'll Master Today

- Define sample spaces and events using proper notation
- Apply probability axioms: $0 \leq P(A) \leq 1$
- Use the complement rule: $P(A') = 1 - P(A)$
- Apply the addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Distinguish between independent and mutually exclusive events
- Solve probability problems in business contexts

. . .

> ! Important
>
> These concepts are fundamental for all probability calculations on the exam!

Part A: Sample Spaces and Events

Random Experiments

A random experiment is a process with uncertain outcomes.

. . .

Examples:

- Rolling a die
- Selecting a product for quality control
- Surveying a customer about satisfaction
- Measuring daily sales

Sample Space

> ! Definition: Sample Space (Ergebnismenge)
>
> The sample space S (or Ω) is the set of all possible outcomes of a random experiment.

…

Examples:

Experiment	Sample Space
Coin flip	$S = \{H, T\}$
Die roll	$S = \{1, 2, 3, 4, 5, 6\}$
Two coin flips	$S = \{HH, HT, TH, TT\}$

Events

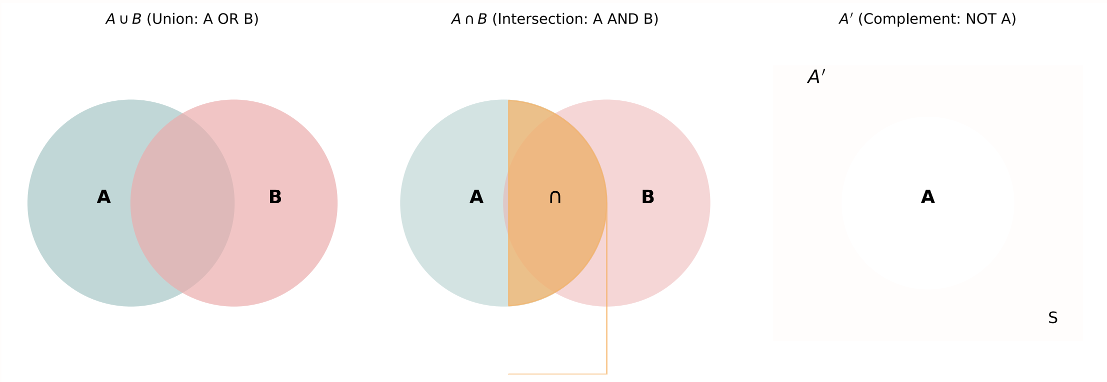
> ! Definition: Event (Ereignis)
>
> An event A is a subset of the sample space S.

…

Example: Die roll with $S = \{1, 2, 3, 4, 5, 6\}$

- Event A: "Rolling an even number" = $\{2, 4, 6\}$
- Event B: "Rolling greater than 4" = $\{5, 6\}$
- Event C: "Rolling a 7" = $\emptyset$ (impossible event)
- Event D: "Rolling a positive number" = S (certain event)

Set Operations on Events



Operation	Notation	Meaning
Union	$A \cup B$	A or B (or both)
Intersection	$A \cap B$	A and B
Complement	A' or $\bar{A}$	Not A

Part B: Probability Axioms

Definition of Probability

! Kolmogorov Axioms

For any event A:

1. $P(A) \geq 0$ (non-negativity)
2. $P(S) = 1$ (certainty)
3. For mutually exclusive events: $P(A \cup B) = P(A) + P(B)$

. . .

Consequence: $0 \leq P(A) \leq 1$ for all events A

Classical Probability

For equally likely outcomes:

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{|A|}{|S|}$$

. . .

Example: Rolling a fair die

$$P(\text{even}) = \frac{|\{2,4,6\}|}{|\{1,2,3,4,5,6\}|} = \frac{3}{6} = \frac{1}{2}$$

Complement Rule

! Complement Rule (Gegenwahrscheinlichkeit)

$$P(A') = 1 - P(A)$$

. . .

Example: If the probability of rain is 0.3, what is the probability of no rain?

$$P(\text{no rain}) = 1 - P(\text{rain}) = 1 - 0.3 = 0.7$$

. . .

Tip

The complement rule is often useful when it's easier to calculate what you don't want!

Example: Using the Complement

A company knows that 5% of its products are defective.

Question: What is the probability that a randomly selected product is NOT defective?

. . .

Solution:

$$P(\text{defective}) = 0.05$$

$$P(\text{not defective}) = 1 - 0.05 = 0.95$$

. . .

Question: In a sample of 3 products, what's the probability that at least one is defective?

. . .

Solution: Use complement!

$$P(\text{at least one defective}) = 1 - P(\text{none defective}) = 1 - (0.95)^3 \approx 0.143$$

Part C: Addition Rule

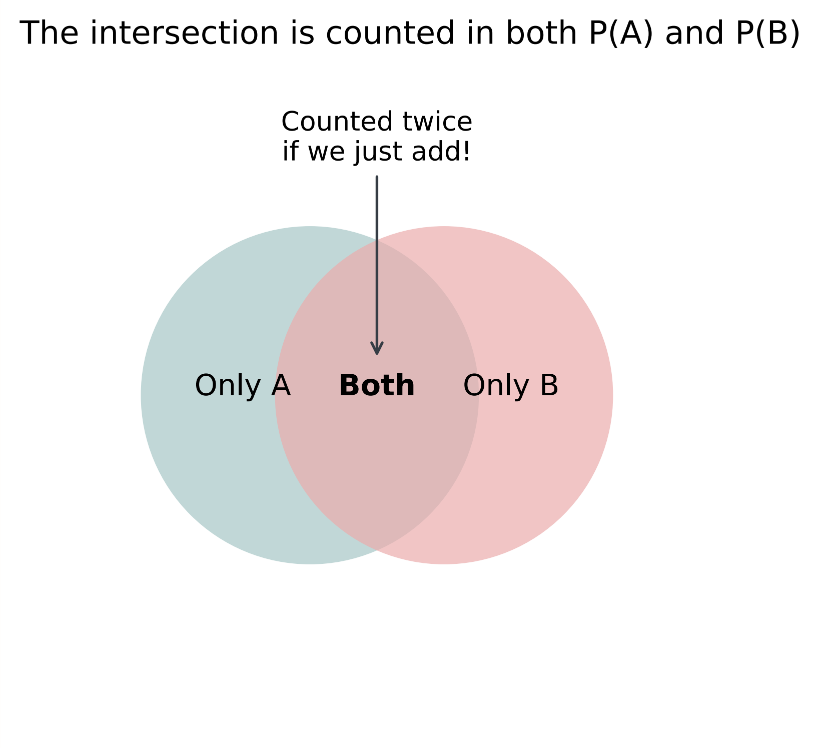
Union of Events

! General Addition Rule (Additionssatz)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

. . .

Why subtract $P(A \cap B)$?



Addition Rule Example

In a class of 100 students:

- 60 study mathematics
- 40 study economics
- 25 study both

. . .

Question: What is the probability that a randomly selected student studies mathematics OR economics?

. . .

Solution:

$$P(M \cup E) = P(M) + P(E) - P(M \cap E)$$

$$= 0.60 + 0.40 - 0.25 = 0.75$$

Part D: Mutually Exclusive Events

Mutually Exclusive (Disjoint) Events

> ! Definition
>
> Events A and B are mutually exclusive (disjunkt) if they cannot occur together:
>
> $$A \cap B = \emptyset \quad \Rightarrow \quad P(A \cap B) = 0$$

. . .

Examples:

- Rolling a 3 and rolling a 5 on one die
- A product being “good” and “defective”
- Being in age group “18-25” and “26-35”

Special Addition Rule

For mutually exclusive events:

$$P(A \cup B) = P(A) + P(B)$$

. . .

Example: Rolling a die, find $P(1 \text{ or } 6)$

. . .

Since rolling 1 and rolling 6 are mutually exclusive:

$$P(1 \cup 6) = P(1) + P(6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Break - 10 Minutes

Part E: Independent Events

Independence

> ! Definition: Independent Events
>
> Events A and B are independent (unabhängig) if the occurrence of one does not affect the probability of the other:
>
> $$P(A \cap B) = P(A) \cdot P(B)$$

. . .

⚠ Warning

Don't confuse: - Mutually exclusive: Can't happen together ($P(A \cap B) = 0$) - Independent: Don't affect each other ($P(A \cap B) = P(A) \cdot P(B)$)

Independence Example

Two machines work independently. Machine A has 95% reliability, Machine B has 90% reliability.

. . .

Question: What is the probability both machines work?

$$P(A \cap B) = P(A) \cdot P(B) = 0.95 \times 0.90 = 0.855$$

. . .

Question: What is the probability at least one machine fails?

$$P(\text{at least one fails}) = 1 - P(\text{both work}) = 1 - 0.855 = 0.145$$

Mutually Exclusive vs Independent

Property	Mutually Exclusive	Independent
$P(A \cap B)$	$= 0$	$= P(A) \cdot P(B)$
Can occur together?	No	Yes
Knowing A occurred…	…tells us B didn't	…tells us nothing about B
Example	"Pass" vs "Fail"	Two separate coin flips

. . .

! Important

If $P(A) > 0$ and $P(B) > 0$, then mutually exclusive events cannot be independent!

Part F: Business Applications

Quality Control Application

A factory produces items with:

- 3% have surface defects (event S)
- 2% have internal defects (event I)
- 0.5% have both defects

...

Find the probability that an item has:

a) At least one type of defect
b) A surface defect but no internal defect
c) Exactly one type of defect

...

Solutions:

a) $P(S \cup I) = P(S) + P(I) - P(S \cap I) = 0.03 + 0.02 - 0.005 = 0.045$

b) $P(S \cap I') = P(S) - P(S \cap I) = 0.03 - 0.005 = 0.025$

c) $P(\text{exactly one}) = P(S \cup I) - P(S \cap I) = 0.045 - 0.005 = 0.04$

Market Research Application

In a survey of 500 consumers:

- 300 prefer Brand A
- 250 prefer organic products
- 150 prefer Brand A AND organic

...

Question: Are "preferring Brand A" and "preferring organic" independent?

...

Check independence:

$$P(A) \cdot P(\text{Org}) = \frac{300}{500} \times \frac{250}{500} = 0.6 \times 0.5 = 0.30$$

$$P(A \cap \text{Org}) = \frac{150}{500} = 0.30$$

...

Since $P(A) \cdot P(\text{Org}) = P(A \cap \text{Org})$, the events are independent!

Guided Practice - 20 Minutes

Practice Problems

Work in pairs

Problem 1: A card is drawn from a standard 52-card deck. a) Find $P(\text{Heart})$ b) Find $P(\text{Face card})$ (J, Q, K) c) Find $P(\text{Heart OR Face card})$

Problem 2: In a company, 40% of employees are in sales, 30% are in engineering, and 10% are in both. Find: a) $P(\text{Sales OR Engineering})$ b) $P(\text{neither Sales nor Engineering})$

Wrap-Up & Key Takeaways

Today's Essential Concepts

- Sample space S: All possible outcomes
- Event: Subset of the sample space
- Probability axioms: $0 \leq P(A) \leq 1$, $P(S) = 1$
- Complement: $P(A') = 1 - P(A)$
- Addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Mutually exclusive: $P(A \cap B) = 0$
- Independent: $P(A \cap B) = P(A) \cdot P(B)$

Next Session Preview

Coming Up: Combinatorics

- Fundamental counting principle
- Permutations: arrangements where order matters
- Combinations: selections where order doesn't matter
- Applications to probability calculations

. . .

> Homework
>
> Complete Tasks 07-02: - Practice sample space identification - Apply probability rules - Solve business probability problems