Session 06-05 - Integration by Parts & Synthesis

Section 06: Integral Calculus

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Entry Quiz - 10 Minutes

Quick Review from Session 06-04

Test your understanding of Area Between Curves & Surplus

1. Find the area between $f(x) = x^2$ and $g(x) = 2x$ from $x = 0$ to $x = 2$.
2. If demand is $D(q) = 80 - 2q$ and supply is $S(q) = 20 + q$, find equilibrium (q^*, p^*).
3. Set up (but don't evaluate) the consumer surplus integral for the market in problem 2.
4. Find $\int (3x^2 - 4x + 5)\, dx$

Homework Discussion - 15 Minutes

Your questions from Session 06-04

Focus on area between curves and economic surplus

- Finding intersection points
- Consumer and producer surplus calculations
- Price floor/ceiling effects on surplus
- Deadweight loss interpretation

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> **ℹ Note**
>
> Today we learn a powerful technique for integrating products of functions and synthesize our integration skills!

Learning Objectives

What You'll Master Today

- Understand integration by parts as the reverse of the product rule
- Apply the LIATE rule to choose which function to differentiate
- Solve integrals of the form $\int x \cdot e^x\, dx$

- Handle repeated application for integrals like $\int x^2 \cdot e^x \, dx$
- Compute average value of a function over an interval
- Apply integration to revenue and cost accumulation
- Synthesize techniques from differential and integral calculus

...

> ! Important
>
> Integration by parts is essential for the Feststellungsprufung!

Part A: The Product Rule in Reverse

Motivation: Why Do We Need This?

Consider this integral:

$$\int x \cdot e^x \, dx$$

...

Question: Can we use the power rule?

...

No!

...

Question: Can we use substitution?

...

Try $u = x$ or $u = e^x$... Neither works cleanly!

...

> ⚠ Warning
>
> Some products of functions require a special technique.

The Product Rule Revisited

Recall the product rule for derivatives:

$$\frac{d}{dx}[u \cdot v] = u' \cdot v + u \cdot v'$$

...

Rearranging:

$$u \cdot v' = \frac{d}{dx}[u \cdot v] - u' \cdot v$$

...

Integrating both sides:

$$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \, dx$$

The Integration by Parts Formula

$$\int \underbrace{u(x)}_{\text{choose}} \cdot \underbrace{v'(x) \, dx}_{= dv} = u(x) \cdot v(x) - \int \underbrace{v(x)}_{\text{from integrating } dv} \cdot \underbrace{u'(x) \, dx}_{= du}$$

...

This is the same as:

$$\int u \, dv = uv - \int v \, du$$

...

Strategy: Choose u and dv so that $\int v \, du$ is easier than $\int u \, dv$!

What Do du and dv Mean?

You already know this notation — from every integral you've written!

- When you write $\int x^2 \, dx$, the dx tells you what variable you integrate with respect to
- Now, du works the same way — it just says “with respect to u”
- The connection between them: if u is a function of x, we can convert between du and dx: $du = u'(x) \, dx$
- In words: du is the derivative of u with respect to x, multiplied by dx
- The dx is needed for the remaining integral, don't worry about it

...

Question: Do you get the basic idea?

Part B: The LIATE Rule

Choosing u - The LIATE Rule

How do we choose which function to call u?

...

Letter	Function Type	Example
L	Logarithmic	$\ln(x)$, $\log(x)$
I	Inverse trig	$\arctan(x)$, $\arcsin(x)$

Letter	Function Type	Example
A	Algebraic	x, x^2, polynomials
T	Trigonometric	$\sin(x)$, $\cos(x)$
E	Exponential	e^x, 2^x

...

Tip

Choose u in this order of priority!

Why LIATE Works

The idea: Choose u to be the function that becomes simpler when differentiated.

- Logarithms ($\ln x \to \frac{1}{x}$): Derivatives are simpler
- Algebraic ($x^n \to nx^{n-1}$): Powers decrease
- Exponentials ($e^x \to e^x$): Don't simplify - put in dv!

...

Note

The LIATE rule is a guideline, not a strict rule. Sometimes you need to experiment!

Part C: Basic Examples

Example 1: $\int x \cdot e^x \, dx$

Step 1: Identify u and dv using LIATE: choose $u = x$ (A comes before E)

...

Step 2: Set up the parts

...

We choose:	We compute:
$u = x$	$du = 1 \cdot dx = dx$ (differentiate u, attach dx)
$dv = e^x \, dx$	$v = \int e^x \, dx = e^x$ (integrate dv to get v)

...

Step 3: Apply the formula $\int u \, dv = uv - \int v \, du$

...

$$\int \underbrace{x}_{u} \cdot \underbrace{e^x\,dx}_{dv} = \underbrace{x}_{u} \cdot \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\,\underbrace{dx}_{du}$$

Example 1: Solution

$$\int x \cdot e^x\,dx = x \cdot e^x - \int e^x\,dx$$

...

$$= x \cdot e^x - e^x + C$$

...

$$= e^x(x-1) + C$$

...

> 💡 Tip
>
> Verification: Differentiate to check:
>
> $$\frac{d}{dx}[e^x(x-1)] = e^x(x-1) + e^x \cdot 1 = e^x \cdot x - e^x + e^x = x \cdot e^x \quad \checkmark$$

Example 2: $\int (x+1) \cdot e^x\,dx$

This is a common exam problem!

...

Step 1: Choose $u = x + 1$ (algebraic), $dv = e^x\,dx$

...

We choose:	We compute:
$u = x + 1$	$du = 1 \cdot dx = dx$ (differentiate u, attach dx)
$dv = e^x\,dx$	$v = \int e^x\,dx = e^x$ (integrate dv to get v)

...

Step 2: Apply the formula

...

$$\int (x+1) \cdot e^x\,dx = (x+1) \cdot e^x - \int e^x\,dx$$

...

$$= (x+1) \cdot e^x - e^x + C = e^x(x+1-1) + C = x \cdot e^x + C$$

Example 3: $\int x \cdot \ln(x)\,dx$

Step 1: Identify using LIATE: choose $u = \ln(x)$ (L comes first!)

...

Step 2: Set up the parts

We choose:	We compute:
$u = \ln(x)$	$du = \frac{1}{x} \cdot dx$ (differentiate u, attach dx)
$dv = x\,dx$	$v = \int x\,dx = \frac{x^2}{2}$ (integrate dv to get v)

...

Step 3: Apply the formula

$$\int x \cdot \ln(x)\,dx = \ln(x) \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx$$

Example 3: Solution

$$\int x \cdot \ln(x)\,dx = \frac{x^2 \ln(x)}{2} - \int \frac{x}{2}\,dx$$

...

$$= \frac{x^2 \ln(x)}{2} - \frac{x^2}{4} + C$$

...

$$= \frac{x^2}{4}(2\ln(x) - 1) + C$$

...

> **i Note**
>
> Notice how choosing $u = \ln(x)$ simplified the integral because $\frac{1}{x} \cdot x^2 = x$ (simpler than before).

Break - 10 Minutes

Part D: Repeated Integration by Parts

When One Application Isn't Enough

Consider: $\int x^2 \cdot e^x\,dx$

...

First application: $u = x^2$, $dv = e^x\,dx$

...

We choose:	We compute:
$u = x^2$	$du = 2x \cdot dx$ (differentiate u, attach dx)
$dv = e^x\,dx$	$v = \int e^x\,dx = e^x$ (integrate dv to get v)

...

$$\int x^2 \cdot e^x\,dx = x^2 \cdot e^x - \int 2x \cdot e^x\,dx$$

...

> Tip
>
> Problem: We still have $\int x \cdot e^x\,dx$ - another integration by parts problem!

Repeated Application

Second application: For $\int x \cdot e^x\,dx$, let $u = x$, $dv = e^x\,dx$

$$\int x \cdot e^x\,dx = x \cdot e^x - e^x$$

...

Substituting back:

$$\int x^2 \cdot e^x\,dx = x^2 \cdot e^x - 2(x \cdot e^x - e^x) + C$$

...

$$= x^2 \cdot e^x - 2x \cdot e^x + 2e^x + C = e^x(x^2 - 2x + 2) + C$$

Exam Example: $\int x^2 \cdot e^{-x}\,dx$

This exact type appeared on the exam!

...

First application: $u = x^2$, $dv = e^{-x}\,dx$, so $v = -e^{-x}$

...

$$\int x^2 \cdot e^{-x}\,dx = -x^2 \cdot e^{-x} - \int -2x \cdot e^{-x}\,dx$$

...

$$= -x^2 \cdot e^{-x} + 2\int x \cdot e^{-x}\,dx$$

...

Second application: $u = x$, $dv = e^{-x}\,dx$, so $v = -e^{-x}$

...

$$\int x \cdot e^{-x}\,dx = -x \cdot e^{-x} + \int e^{-x}\,dx = -x \cdot e^{-x} - e^{-x}$$

Exam Example: Final Answer

Substituting back:

$$\int x^2 \cdot e^{-x}\,dx = -x^2 \cdot e^{-x} + 2(-x \cdot e^{-x} - e^{-x}) + C$$

...

$$= -x^2 \cdot e^{-x} - 2x \cdot e^{-x} - 2e^{-x} + C$$

...

$$= -e^{-x}(x^2 + 2x + 2) + C$$

...

> ! Important
>
> Verification is important! Differentiate your answer if you have the time:
>
> $$\frac{d}{dx}[-e^{-x}(x^2 + 2x + 2)] = e^{-x}(x^2 + 2x + 2 - 2x - 2) = x^2 e^{-x}$$

Part E: Definite Integrals by Parts

Definite Integrals with Bounds

For definite integrals:

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

...

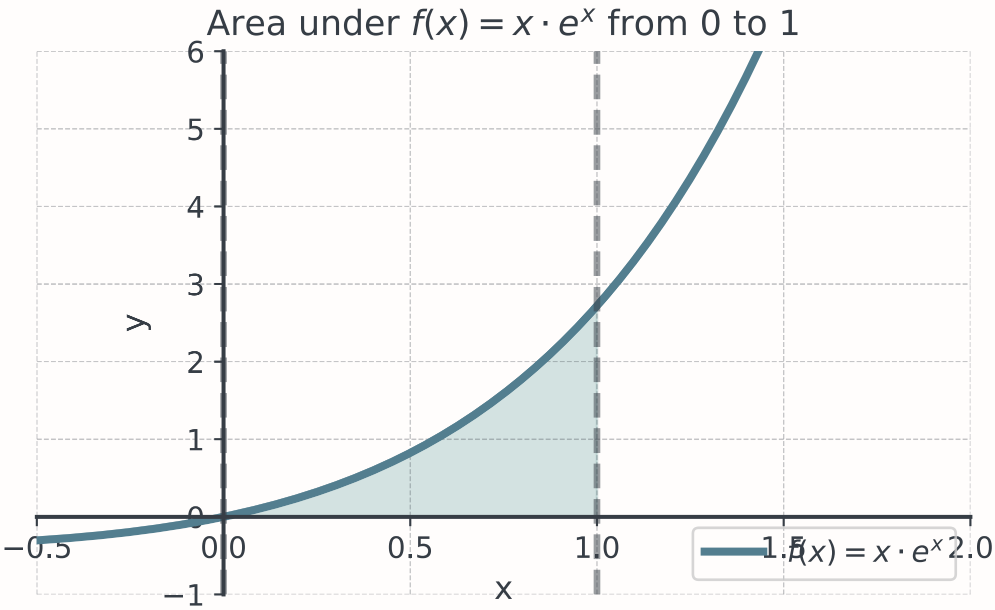
Example: $\int_0^1 x \cdot e^x\,dx$, we found $\int x \cdot e^x\,dx = e^x(x-1) + C$

...

$$\int_0^1 x \cdot e^x\,dx = [e^x(x-1)]_0^1 = e^1(1-1) - e^0(0-1)$$

$$= 0 - (-1) = 1$$

Definite Integral Example with Graph



...

> ! Important
>
> Essentially, we use integration by parts and then compute the definite integral as learned.

Guided Practice - 20 Minutes

Practice Set A: Single Application

Work individually for 5 minutes

1. $\int 3x \cdot e^x \, dx$
2. $\int (2x + 1) \cdot e^x \, dx$
3. $\int x \cdot e^{-x} \, dx$
4. $\int x^2 \cdot \ln(x) \, dx$

Practice Set B: Repeated Application

Work individually for 7 minutes

1. $\int x^2 \cdot e^{2x} \, dx$
2. $\int (x^2 + x) \cdot e^x \, dx$
3. $\int_0^2 x \cdot e^x \, dx$ (evaluate with bounds)

Coffee Break - 15 Minutes

Part F: Average Value of a Function

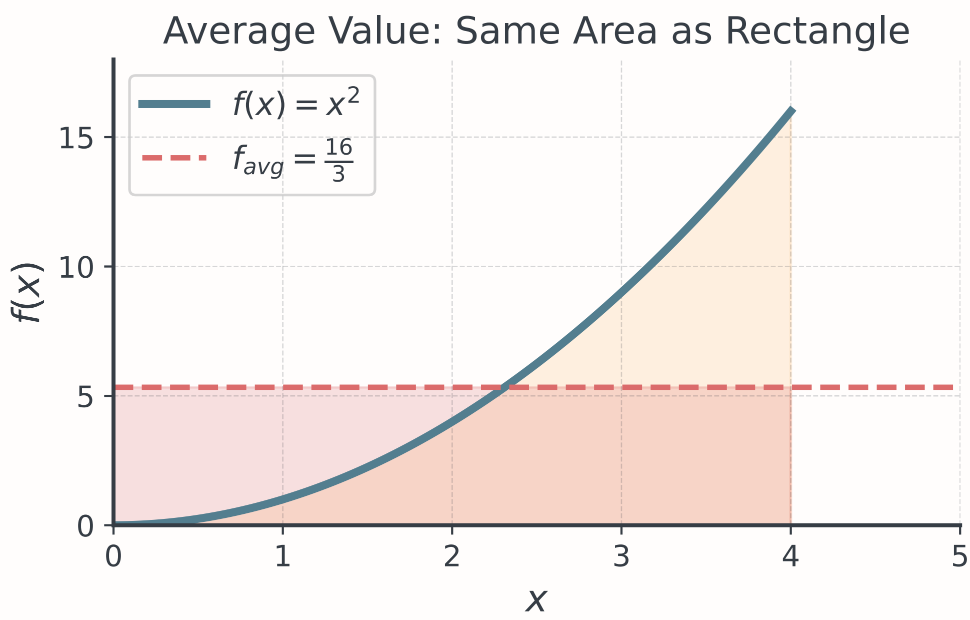
The Average Value Formula

Key idea: What is the "average height" of a function over an interval?

...

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

The Average Visualized



...

> Tip
>
> Makes sense, doesn't it?

Example: Average Value

Find the average value of $f(x) = x^2$ on $[0, 4]$

...

$$f_{avg} = \frac{1}{4-0} \int_0^4 x^2\, dx$$

...

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

...

$$= \frac{1}{4} \cdot \frac{64}{3} = \frac{16}{3} \approx 5.33$$

...

> 💡 Tip
>
> The area under $f(x)$ equals the area of a rectangle with height f_{avg} and width $(b - a)$.

Part G: Revenue and Cost Accumulation

Total Revenue from Marginal Revenue

Key relationship: If $MR(x)$ is the marginal revenue, then total revenue from selling a to b units is:

$$\text{Revenue}_{a \to b} = \int_a^b MR(x)\, dx$$

...

Example: $MR(x) = 100 - 2x$, revenue from selling units 10 to 30:

...

$$\int_{10}^{30} (100 - 2x)\, dx = [100x - x^2]_{10}^{30}$$

...

$$= (3000 - 900) - (1000 - 100) = 2100 - 900 = 1200 \quad \text{EUR}$$

Profit Accumulation Over Time

Scenario: Profit rate $P'(t) = 8t - t^2$ thousand EUR per month (t)

Find total profit from month 2 to month 6.

...

$$P_{2 \to 6} = \int_2^6 (8t - t^2)\, dt = \left[4t^2 - \frac{t^3}{3}\right]_2^6$$

...

$$= (144 - 72) - \left(16 - \frac{8}{3}\right)$$

...

$$= 72 - \frac{40}{3} = \frac{216 - 40}{3} = \frac{176}{3} \approx 58.67 \quad \text{thousand EUR}$$

Collaborative Problem-Solving - 25 Minutes

Practice Set C: Synthesis

Work with a partner for 10 minutes

1. Find the average value of $f(x) = 2x + 1$ on $[0, 4]$
2. A company's marginal profit is $P'(x) = 80 - 4x$ EUR/unit.
   a) Find the production level x^* that maximizes profit.
   b) Calculate the total profit from $x = 5$ to $x = 15$.
3. A machine's production rate is $P(t) = 100e^{-0.1t}$ units/hour. Find the average production rate over the first 10 hours.

Challenge: Present Value Analysis

Scenario: A company expects future profits that decay over time. The profit rate is the follwoing where t is years and profit is in thousands of euros.

$$P'(t) = (100 + 20t) \cdot e^{-0.05t}$$

a) Find the antiderivative of the profit rate using integration by parts

b) Calculate the total profit from year 0 to year 10

c) At what rate is profit changing at $t = 5$?

d) Interpret the results for a business decision

Wrap-Up & Key Takeaways

What We've Learned in Section 06

1. Antiderivatives reverse differentiation: $F'(x) = f(x)$
2. Definite integrals compute net accumulated change
3. Area under curves using $\int_a^b f(x)\,dx$
4. Area between curves using $\int_a^b [f(x) - g(x)]\,dx$
5. Consumer Surplus: $CS = \int_0^{q^*} [D(q) - p^*]\,dq$
6. Producer Surplus: $PS = \int_0^{q^*} [p^* - S(q)]\,dq$
7. Integration by Parts: $\int u\,dv = uv - \int v\,du$
8. Average Value: $f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$

Essential Formulas Summary

Concept	Formula
Antiderivative	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$
FTC	$\int_a^b f(x) \, dx = F(b) - F(a)$
Area Between	$\int_a^b [f(x) - g(x)] \, dx$
Consumer Surplus	$\int_0^{q^*} [D(q) - p^*] \, dq$
Producer Surplus	$\int_0^{q^*} [p^* - S(q)] \, dq$
Integration by Parts	$\int u \, dv = uv - \int v \, du$
Average Value	$\frac{1}{b-a} \int_a^b f(x) \, dx$

Final Assessment - 5 Minutes

Quick Check

Work individually

1. Find $\int (x + 2) \cdot e^x \, dx$
2. Find the average value of $f(x) = x^2$ on $[1, 3]$.
3. Why does LIATE put logarithms before algebraic functions?
4. What does consumer surplus measure economically?

Next Session Preview

Session 06-06: Mock Exam

Comprehensive Assessment

- Full mock exam covering differential and integral calculus
- 180 minutes working time with problems covering:
  - Graphical differentiation and integration
  - Function analysis with calculus
  - Area calculations
  - Economic applications
  - Integration by parts

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> ! Important
>
> Complete tasks and review all integration techniques!