# Session 06-03 - Area Problems & Basic Applications

## Section 06: Integral Calculus

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# Entry Quiz - 10 Minutes

## Quick Review from Session 06-02

Test your understanding of Definite Integrals

1. Evaluate $\int_1^4 (3x^2 - 2x)\,dx$
2. If $\int_0^5 f(x)\,dx = 18$ and $\int_3^5 f(x)\,dx = 7$, find $\int_0^3 f(x)\,dx$
3. What does $\int_a^b f'(x)\,dx$ represent geometrically and algebraically?
4. For $f(x) = x - 2$ on $[0, 4]$, is the signed area positive, negative, or zero?

# Homework Discussion - 15 Minutes

## Your questions from Session 06-02

Focus on FTC and definite integrals

- Evaluating definite integrals with bounds
- Signed area vs. total area distinction
- Properties of definite integrals
- Net change applications

...

> **ℹ Note**
>
> Today we focus on area calculations and introduce exponential and logarithmic integrals!

# Learning Objectives

## What You'll Master Today

- Calculate area under a curve above the x-axis
- Handle regions where the function is below the x-axis
- Find total area by splitting at zeros
- Integrate exponential functions $\int e^{ax}\,dx$

- Integrate $\frac{1}{x}$ to get natural logarithm
- Apply area concepts to business problems
- Interpret accumulated quantities from rate functions

...

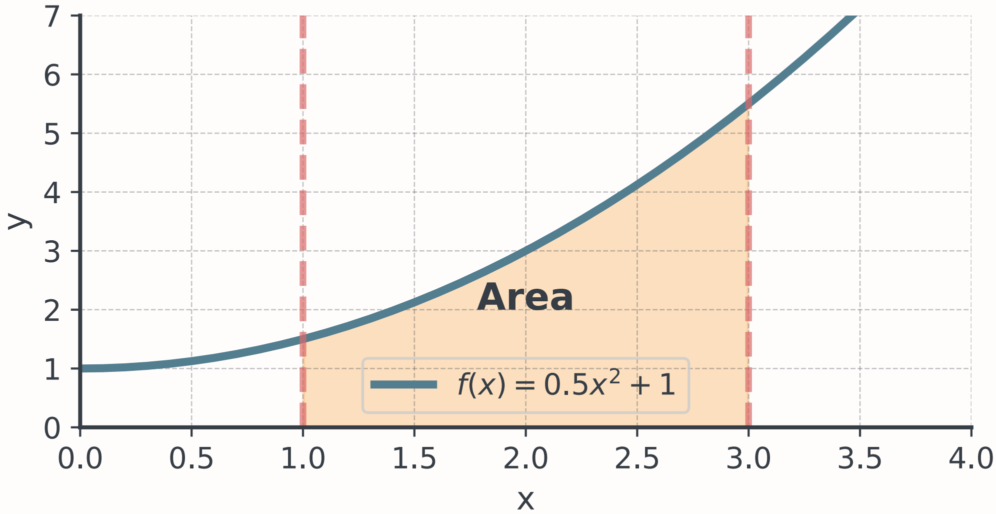
> **i Note**
>
> Area calculations are one of the most common applications of integration!

## Part A: Area Under a Curve

### When $f(x) \geq 0$

Simple case: When $f(x) \geq 0$ on $[a, b]$, the definite integral gives the area directly.

$$\text{Area} = \int_a^b f(x)\, dx$$




### Example: Area Under a Parabola I

Find the area under $f(x) = x^2$ from $x = 0$ to $x = 3$.

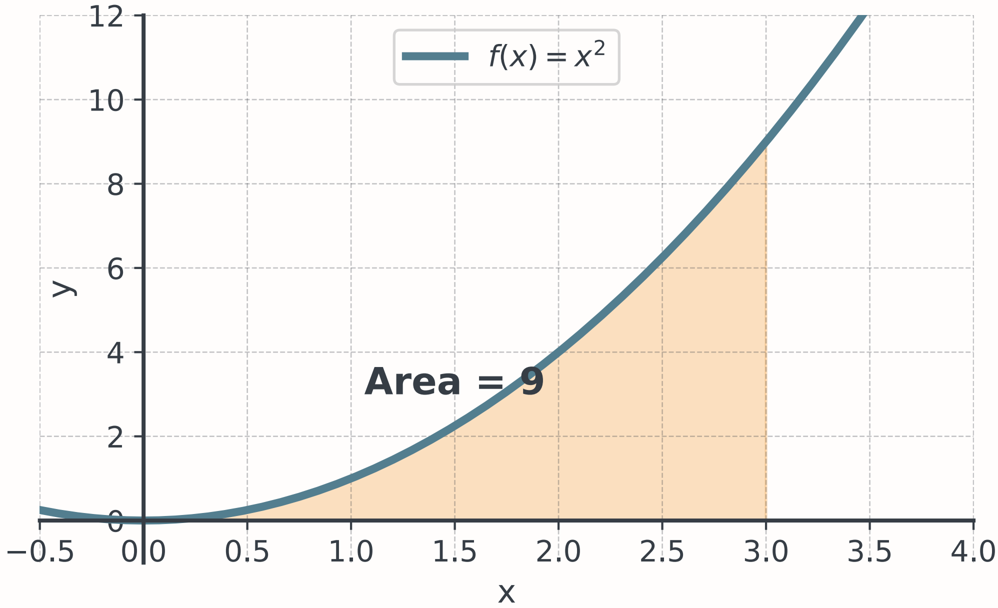
...

Solution:

$$\text{Area} = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3 = \frac{27}{3} - 0 = 9$$

### Example: Area Under a Parabola II

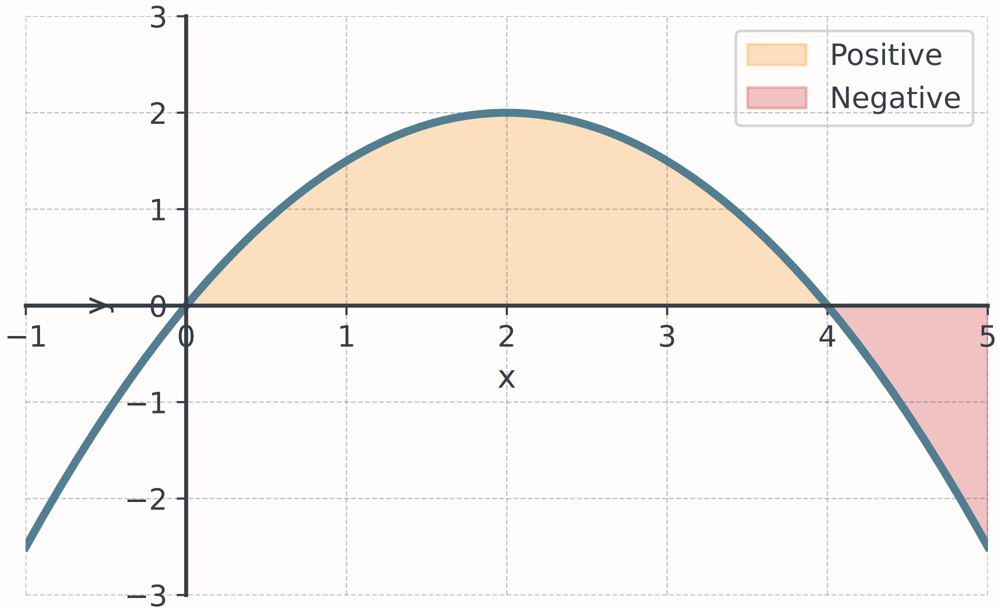
Find the area under $f(x) = x^2$ from $x = 0$ to $x = 3$.




## Part B: Area When $f(x) < 0$

### The Sign Problem

When $f(x) < 0$: The definite integral gives a negative value!

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> ⚠ Warning
>
> Definite integral ≠ Total area when function crosses the x-axis!

## Total Area Strategy

To find total (unsigned) area:

1. Find where $f(x) = 0$ (zeros/roots)
2. Split the integral at each zero
3. Take absolute value of each piece
4. Add all the positive values

...

$$\text{Total Area} = \sum |\text{each region}|$$

## Example: Finding Total Area

Total area between $f(x) = x^2 - 4$ and x-axis from $x = 0$ to $x = 3$?

...

Step 1: Find zeros: $x^2 - 4 = 0 \implies x = \pm 2$

- Only $x = 2$ is in $[0, 3]$.

...

Step 2: Determine signs:

- $f(1) = -3 < 0$ (below x-axis on $[0, 2]$)
- $f(3) = 5 > 0$ (above x-axis on $[2, 3]$)

## Completing the Calculation

Step 3: Calculate each piece:

$$\int_0^2 (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_0^2 = \frac{8}{3} - 8 = -\frac{16}{3}$$

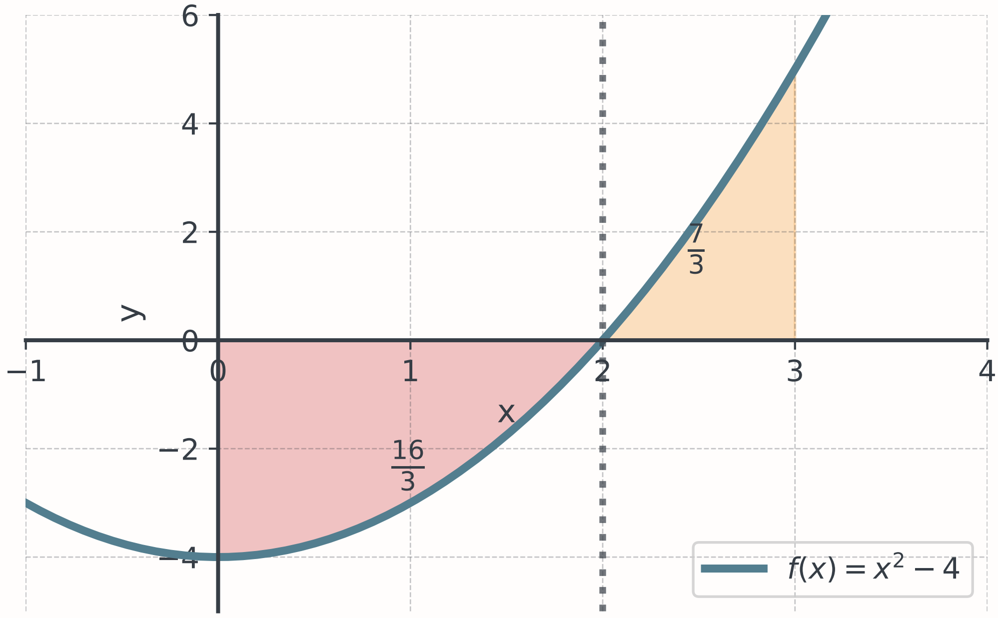
$$\int_2^3 (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_2^3 = (9 - 12) - \left(\frac{8}{3} - 8\right) = -3 + \frac{16}{3} = \frac{7}{3}$$

...

Step 4: Total area:

$$\text{Total Area} = \left|-\frac{16}{3}\right| + \frac{7}{3} = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$$

Visualization

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Total Area = $\frac{16}{3} + \frac{7}{3} = \frac{23}{3} \approx 7.67$

## Break - 10 Minutes

## Part C: Exponential Integrals

### Integrating $e^x$

Recall: $\frac{d}{dx}[e^x] = e^x$

...

Therefore:

$$\int e^x \, dx = e^x + C$$

...

> Tip
>
> The exponential function is its own antiderivative!

### Integrating $e^{ax}$

For $e^{ax}$ where $a$ is a constant:

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

...

Verification: $\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = \frac{1}{a} \cdot a \cdot e^{ax} = e^{ax}$ ✓

...

**i Note**

Trick: Divide by the coefficient of $x$ in the exponent.

### Examples: Exponential Integrals

Example 1: $\int e^{3x}\,dx$

...

- $\frac{1}{3}e^{3x} + C$

...

Example 2: $\int e^{-2x}\,dx$

...

- $\frac{1}{-2}e^{-2x} + C = -\frac{1}{2}e^{-2x} + C$

...

Example 3: $\int 4e^{5x}\,dx$

...

- $4 \cdot \frac{1}{5}e^{5x} + C = \frac{4}{5}e^{5x} + C$

## Part D: Logarithmic Integral

### The Missing Case: $n = -1$

Recall the power rule:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for} \quad n \neq -1$$

...

Why not $n = -1$? Division by zero!

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = ???$$

...

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

## Why the Absolute Value?

- Recall: $\frac{d}{dx}[\ln x] = \frac{1}{x}$ (for $x > 0$)
- But what about $x < 0$?
- For $x < 0$: $\frac{d}{dx}[\ln(-x)] = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

. . .

> ! Important
>
> $$\int \frac{1}{x}\,dx = \ln|x| + C$$
>
> The absolute value handles both positive and negative $x$.

## Examples: Logarithmic Integrals

Example 1: $\int_1^e \frac{1}{x}\,dx$

. . .

- $\ln e - \ln 1 = 1 - 0 = 1$

. . .

Example 2: $\int_1^4 \frac{3}{x}\,dx$

. . .

- $3(\ln 4 - \ln 1) = 3\ln 4$

. . .

Example 3: $\int_{-3}^{-1} \frac{2}{x}\,dx$

. . .

- $2(\ln 1 - \ln 3) = -2\ln 3$

## Summary: Special Integrals

| Function | Antiderivative |
|---|---|
| $e^x$ | $e^x + C$ |
| $e^{ax}$ | $\frac{1}{a}e^{ax} + C$ |
| $\frac{1}{x}$ | $\ln\|x\| + C$ |
| $\frac{1}{ax+b}$ | $\frac{1}{a}\ln\|ax+b\| + C$ |

. . .

Tip

These formulas will appear frequently in business applications!

# Guided Practice - 20 Minutes

## Set A: Area Calculations

Work individually for 8 minutes

1. Find the area under $f(x) = 3x^2$ from $x = 1$ to $x = 4$.
2. Find the total area between $f(x) = x - 3$ and the x-axis from $x = 0$ to $x = 5$.
3. Find the total area between $f(x) = x^2 - 1$ and the x-axis from $x = 0$ to $x = 2$.

## Set B: Exponential & Logarithmic

Work individually for 6 minutes

1. $\int e^{4x}\,dx$
2. $\int 5e^{-x}\,dx$
3. $\int_0^2 e^{3x}\,dx$
4. $\int_1^5 \frac{2}{x}\,dx$
5. $\int \left(e^x + \frac{1}{x}\right)dx$

## Practice Set C: Mixed Problems

Work in pairs for 6 minutes

1. Find the area enclosed between $f(x) = e^x$ and the x-axis from $x = 0$ to $x = 2$.
2. A population decays according to $P(t) = 1000e^{-0.1t}$. Find the average population from $t = 0$ to $t = 10$. (Hint: Average = $\frac{1}{b-a}\int_a^b f(x)\,dx$)

# Coffee Break - 15 Minutes

# Part E: Business Applications

## Total Profit Over Time

Scenario: A company's profit rate (profit per month) is:

$$P'(t) = 50 - 2t \quad \text{thousand euros per month}$$

where $t$ is months since launch.

...

Questions:

1. What is the total profit during the first year ($t = 0$ to $t = 12$)?
2. At what month does profit rate become negative?

## Solution: Profit Analysis

Part 1: Total profit

$$\int_0^{12} (50 - 2t)\, dt = \left[50t - t^2\right]_0^{12} = 600 - 144 = 456$$

Total profit = €456,000

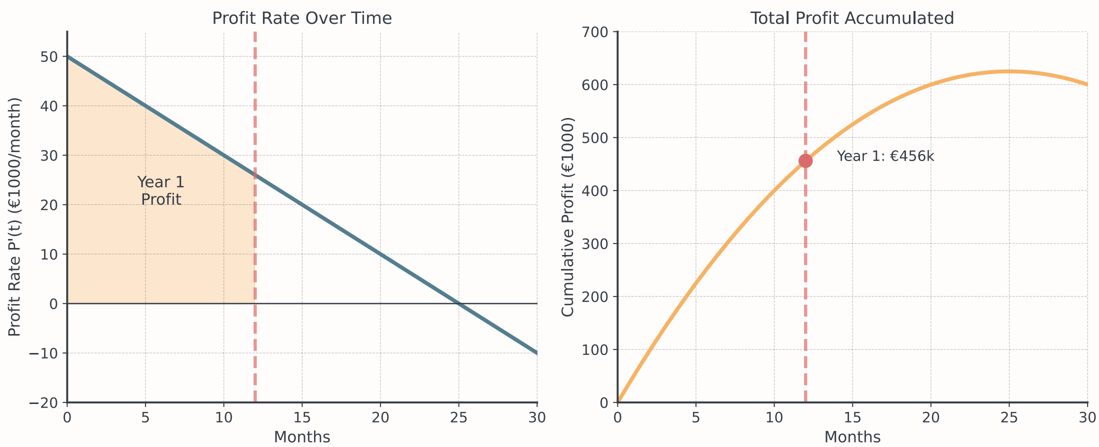
...

Part 2: When profit rate becomes zero

$$50 - 2t = 0 \implies t = 25 \quad \text{months}$$

The profit rate stays positive for the first 25 months.

## Visualizing Profit Accumulation




## Exponential Decay in Business

Scenario: Sales of a product decline exponentially after its peak:

$$S(t) = 10000 \cdot e^{-0.2t} \quad \text{units per month}$$

...

Question: What are the total sales from $t = 0$ to $t = 6$ months?

...

Solution:

$$\int_0^{6} 10000e^{-0.2t}\, dt = 10000 \cdot \frac{1}{-0.2} e^{-0.2t}\Big|_0^6$$

$$= -50000(e^{-1.2} - e^0) = -50000(0.301 - 1) = 34,950 \quad \text{units}$$

# Collaborative Problem-Solving - 30 Minutes

## Group Challenge: Market Analysis

Scenario: An e-commerce company tracks its daily revenue rate:

$$R'(t) = 5000 + 200t - 5t^2 \quad \text{euros per day}$$

where $t$ is days since a marketing campaign started.

The campaign runs for 30 days.

## Group Tasks

Work in groups of 3-4

1. Graph $R'(t)$ for the 30-day period. When is the revenue rate highest?
2. Calculate the total revenue for the first 10 days.
3. Calculate the total revenue for the entire 30-day campaign.
4. On which day does the revenue rate first drop below €4,000/day?
5. Find the average daily revenue rate over the 30-day campaign.
6. If the campaign costs €80,000, what is the net profit?

# Wrap-Up & Key Takeaways

## Today's Essential Concepts

- Area under curve when $f(x) \geq 0$: Use $\int_a^b f(x)\,dx$ directly
- Total area: Split at zeros and sum absolute values
- Exponential integrals: $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
- Logarithmic integral: $\int \frac{1}{x}\,dx = \ln|x| + C$
- Business applications: Total quantities from rate functions
- Average value: $\frac{1}{b-a}\int_a^b f(x)\,dx$

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> ! Important
>
> Next session: Area between TWO curves and economic surplus!

# Final Assessment - 5 Minutes

## Quick Check

Work individually, then compare

1. Find the total area between $f(x) = x - 2$ and the x-axis from $x = 0$ to $x = 4$.

2. Evaluate $\int_0^3 2e^{-x}\, dx$.
3. A company's revenue rate is $R'(t) = 100 + 20t$ thousand euros per month. Find total revenue for months 1-5.

## Next Session Preview

### Coming Up: Area Between Curves

- Finding intersection points of two functions
- Determining which function is "on top"
- Setting up $\int_a^b [f(x) - g(x)]\, dx$
- Handling multiple regions
- Consumer and producer surplus introduction

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> Tip
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> Complete Tasks 06-03
>
> - Practice area calculations with sign changes
> - Work with exponential and logarithmic integrals
> - Focus on business rate-to-total problems