Session 06-02 - Definite Integrals & The Fundamental Theorem

Section 06: Integral Calculus

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Entry Quiz - 10 Minutes

Quick Review from Session 06-01

Test your understanding of Antiderivatives

1. Find $\int (4x^3 - 6x + 2)\, dx$
2. Find $\int \frac{3}{x^2}\, dx$
3. If $f'(x) = 2x + 1$ and $f(0) = 3$, find $f(x)$.
4. Why do we write $+C$ when finding indefinite integrals?

Homework Discussion - 15 Minutes

Your questions from Session 06-01

Focus on antiderivatives and indefinite integrals

- Challenges with the power rule
- Initial value problems
- Business applications (marginal cost → total cost)
- Verification by differentiation

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> **i Note**
>
> Today we connect integration to area and discover the Fundamental Theorem of Calculus!

Learning Objectives

What You'll Master Today

- Understand the definite integral as a limit of sums
- Interpret definite integrals as signed area under curves
- Apply the Fundamental Theorem of Calculus to evaluate integrals

- Use the evaluation formula $\int_a^b f(x)\,dx = F(b) - F(a)$
- Apply properties of definite integrals
- Interpret net change using definite integrals
- Connect integration to accumulation in business contexts

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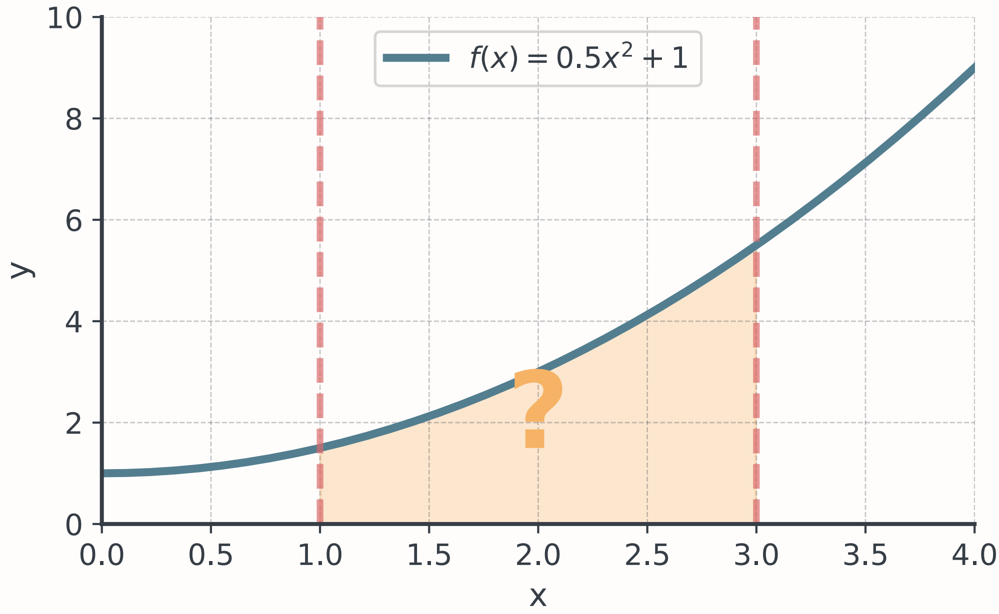
ℹ Note

The Fundamental Theorem connects the area problem to antiderivatives!

Part A: The Area Problem

Motivating Question

How do we find the area under a curve?



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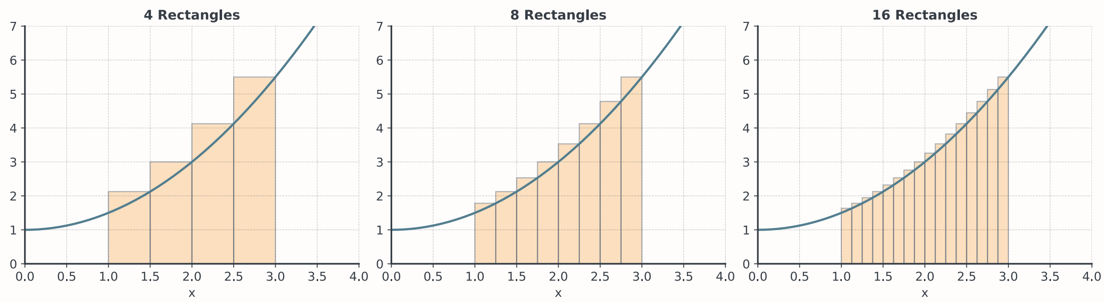
Tip

Question: What is the area of the shaded region between $x = 1$ and $x = 3$?

The Rectangle Approximation Idea

Strategy: Approximate the curved region with rectangles!



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More rectangles ⟶ Better approximation!

The Riemann Sum

Definition: A Riemann sum approximates area using rectangles:

$$\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

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- $\Delta x = \frac{b-a}{n}$ is the width of each rectangle
- x_i^* is a sample point in the i-th subinterval
- $f(x_i^*)$ is the height of the i-th rectangle
- We sum the areas of all n rectangles
- As $n \to \infty$, the Riemann sum approaches the exact area!

From Sum to Integral

The Definite Integral:

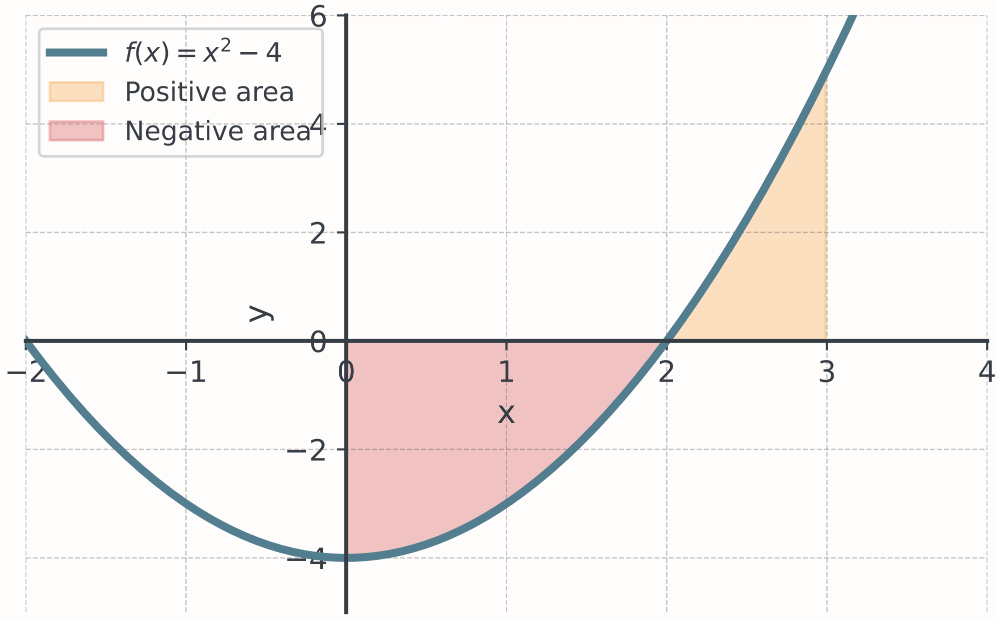
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

- $\int_a^b$ means "integrate from a to b"
- a is the lower limit of integration
- b is the upper limit of integration
- The result is a number (not a function!)

Part B: Signed Area

Area Above vs. Below the x-axis

Key insight: Definite integrals measure signed area!



Signed Area Rules

The definite integral gives signed area:

- Above x-axis: Area counts as positive
- Below x-axis: Area counts as negative
- Total: Positive area minus negative area

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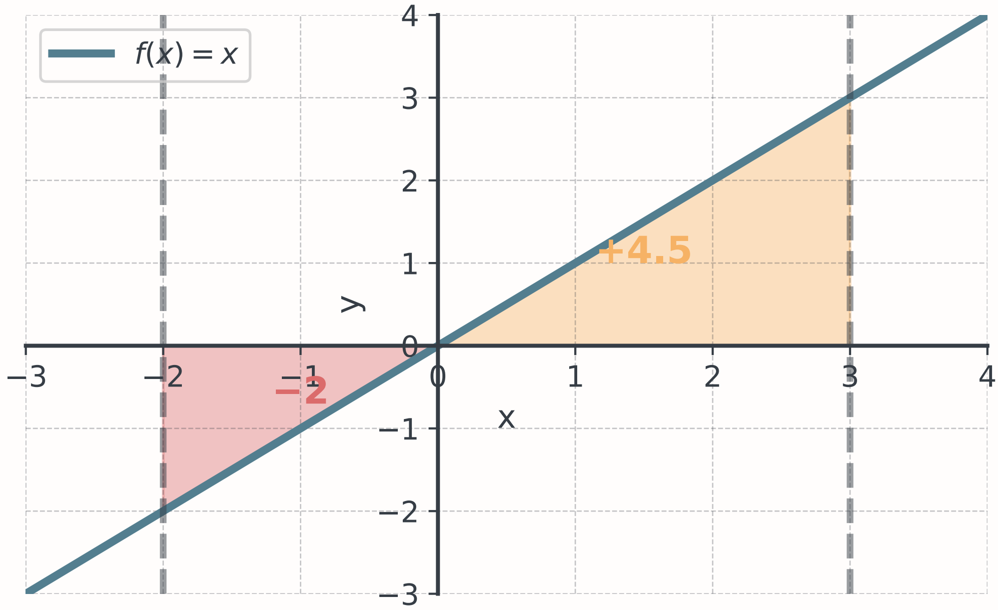
⚠ Warning

Signed area ≠ Total area

If you want total (unsigned) area, you need to handle regions above and below separately!

Example: Signed vs. Total Area



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- Signed area: $\int_{-2}^{3} x\,dx = 4.5 - 2 = 2.5$
- Total area: $|-2| + 4.5 = 6.5$

Break - 10 Minutes

Part C: The Fundamental Theorem of Calculus

The Big Connection

The Fundamental Theorem of Calculus (FTC):

. . .

If $F(x)$ is an antiderivative of $f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

. . .

> ! Important
>
> This is one of the most important theorems in mathematics! It connects:
>
> - Differentiation (finding rates of change)
> - Integration (finding accumulated quantities/areas)

Why the FTC Works

Intuition: The integral accumulates the rate of change.

- $f(x)$ represents a rate of change
- $F(x)$ represents the total accumulated quantity
- $F(b) - F(a)$ = total change from $x = a$ to $x = b$

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> **ℹ Note**
>
> Imagine filling a water tank where $f(x)$ is the flow rate (liters per minute).
>
> - $F(x)$ is the water level at time x
> - So $F(b) - F(a)$ = (final level) – (initial level) = total liters added!
> - Each tiny change $f(x) \cdot \Delta x$ adds up to $F(b) - F(a)$.

Notation: The Evaluation Bar

We write:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

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The notation $F(x)\Big|_a^b$ means "evaluate F at b and subtract F at a"

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Also written as:

$$[F(x)]_a^b = F(b) - F(a)$$

Example 1: Basic Evaluation

Evaluate: $\int_1^4 2x\,dx$

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Step 1: Find an antiderivative

$$F(x) = x^2$$

...

Step 2: Apply the FTC

$$\int_1^4 2x\,dx = x^2\Big|_1^4 = 4^2 - 1^2 = 16 - 1 = 15$$

...

! Important

No $+C$ needed for definite integrals! The constants cancel: $(F(b)+C)-(F(a)+C)=F(b)-F(a)$

Example 2: Polynomial

Evaluate: $\int_0^3 (x^2-2x+1)\,dx$

...

Step 1: Find an antiderivative

...

$$F(x)=\frac{x^3}{3}-x^2+x$$

...

Step 2: Evaluate at bounds

...

$$=\left[\frac{x^3}{3}-x^2+x\right]_0^3=\left(\frac{27}{3}-9+3\right)-(0-0+0)$$

...

$$=9-9+3=3$$

Example 3: With Negative Values

Evaluate: $\int_{-1}^2 3x^2\,dx$

...

Step 1: Antiderivative is $F(x)=x^3$

...

Step 2: Evaluate

$$\int_{-1}^2 3x^2\,dx=x^3\Big|_{-1}^2=2^3-(-1)^3=8-(-1)=9$$

...

i Note

Be careful with negative lower limits! $(-1)^3=-1$, not 1.

Part D: Properties of Definite Integrals

Key Properties

Property 1: Reversing Limits

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

...

Property 2: Splitting Intervals

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

More Properties

Property 3: Constant Multiple

$$\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$$

...

Property 4: Sum/Difference

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

...

> Tip
>
> These properties are the same as for indefinite integrals, just applied to definite integrals!

Using Properties: Example

Given: $\int_0^5 f(x)\,dx = 12$ and $\int_0^3 f(x)\,dx = 7$, find: $\int_3^5 f(x)\,dx$

...

Solution: Using the splitting property:

$$\int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx = \int_0^5 f(x)\,dx$$

...

$$7 + \int_3^5 f(x)\,dx = 12$$

...

$$\int_3^5 f(x)\,dx = 5$$

Guided Practice - 20 Minutes

Practice Set A: Basic Evaluation

Work individually for 5 minutes

Evaluate these definite integrals:

1. $\int_0^2 3x^2\,dx$
2. $\int_1^4 (2x-1)\,dx$
3. $\int_{-2}^1 x^3\,dx$
4. $\int_0^1 (x^2+x+1)\,dx$

Practice Set B: More Complex

Work individually for 7 minutes

1. $\int_1^9 \frac{2}{\sqrt{x}}\,dx$
2. $\int_{-1}^3 (4-x^2)\,dx$
3. Given $\int_0^6 g(x)\,dx = 15$ and $\int_4^6 g(x)\,dx = 8$, find $\int_0^4 g(x)\,dx$
4. $\int_2^5 (3x^2-4x+2)\,dx$

Practice Set C: Area Interpretation

Work in pairs for 8 minutes

1. Calculate the signed area: $\int_{-2}^2 x^3\,dx$. Explain why this result makes geometric sense.
2. For $f(x) = x-1$ from $x=0$ to $x=3$:
   a) Calculate the signed area $\int_0^3 (x-1)\,dx$
   b) Find the total (unsigned) area between the curve and x-axis

Coffee Break - 15 Minutes

Part E: The Net Change Theorem

Net Change Interpretation

The Fundamental Theorem as Net Change:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

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In words: The integral of a rate of change gives the net change in the original quantity.

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Note

Rate (Derivative)	Integral Gives
Marginal cost $C'(x)$	Change in cost $C(b) - C(a)$
Population growth rate	Change in population
Production rate	Total production

Business Example: Total Cost Change

Scenario: A company's marginal cost is $MC(x) = C'(x) = 20 + 0.5x$ euros per unit. What is the total cost of increasing production from 10 to 50 units?

. . .

Solution:

$$\int_{10}^{50} (20 + 0.5x)\, dx = [20x + 0.25x^2]_{10}^{50}$$

. . .

$$= (1000 + 625) - (200 + 25) = 1625 - 225 = €1400$$

. . .

Tip

The total additional cost of producing 40 more units is €1,400.

Business Example: Monthly Profit Rate

Scenario: A startup's monthly profit rate (in thousands of euros) is:

$$P'(t) = t^2 - 4t + 3 \quad \text{(thousands €/month)}$$

where t is months since launch.

. . .

Question: What is the net profit over the first 4 months?

. . .

Solution:

$$\int_0^4 (t^2 - 4t + 3)\, dt = \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^4 = \frac{4}{3} \text{ thousand} = €1,333$$

Net Profit vs. Total Cash Flow

- Net profit = $\int_a^b P'(t)\, dt$ (profits minus losses)
- Total cash movement = $\int_a^b |P'(t)|\, dt$ (all money that moved)

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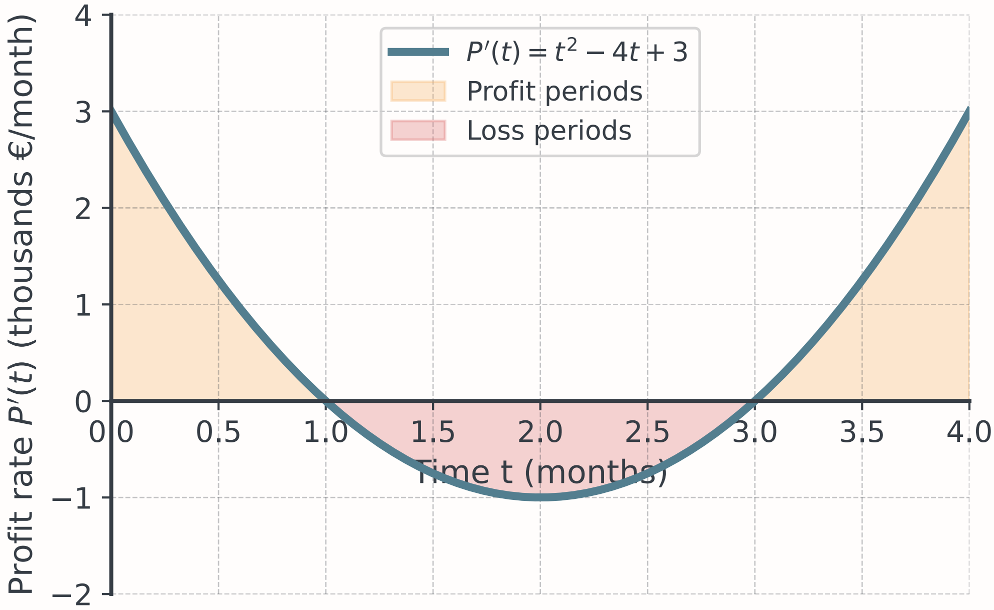

Why Both Measures Matter

Business interpretation:

- Months 0-1: Profitable (early adopters)
- Months 1-3: Losses (scaling costs exceed revenue)
- Months 3-4: Profitable again (economies of scale kick in)

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> ⚠ Warning
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> For investors: Net profit of €1,333 sounds good!

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> ⚠ Warning
>
> For cash management: The company needed reserves to survive the loss period (months 1-3), even though it ended up profitable overall.

Collaborative Problem-Solving - 30 Minutes

Group Challenge: Production Analysis

Scenario: A manufacturing plant's production rate varies during an 8-hour shift:

$$P(t) = 50 + 30t - 3t^2 \quad \text{units per hour}$$

where t is hours since the shift started.

Group Tasks

Work in groups of 3-4

1. Graph $P(t)$ for $0 \leq t \leq 8$. At what time is production rate highest?
2. Calculate the total production during the first 4 hours: $\int_0^4 P(t)\,dt$
3. Calculate total production for the entire 8-hour shift.
4. When does production rate drop to zero? What does this mean?
5. If workers are paid €0.50 per unit, calculate total labor cost for:
   a) The first half of the shift (0-4 hours)
   b) The second half (4-8 hours)

Wrap-Up & Key Takeaways

Today's Essential Concepts

- Definite integral = limit of Riemann sums = signed area
- Fundamental Theorem: $\int_a^b f(x)\,dx = F(b) - F(a)$
- Signed area: Above x-axis positive, below negative
- Properties: Reversing limits, splitting intervals, linearity
- Net change: $\int_a^b f'(x)\,dx = f(b) - f(a)$
- Applications: Total cost, displacement, accumulated quantities

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> ! Important
>
> Next session: Area under curves and between curves with applications!

Comparison: Indefinite vs. Definite

Indefinite Integral	Definite Integral
$\int f(x)\,dx$	$\int_a^b f(x)\,dx$
Result is a function + C	Result is a number
Family of antiderivatives	Signed area / net change
Need $+C$	No $+C$ needed

Final Assessment - 5 Minutes

Quick Check

Work individually, then compare

1. Evaluate $\int_1^3 (2x+4)\,dx$
2. If $\int_0^5 f(x)\,dx = 20$ and $\int_0^2 f(x)\,dx = 6$, find $\int_2^5 f(x)\,dx$
3. A company's marginal profit is $MP(x) = 100 - 2x$. Find the total profit gained by increasing production from 20 to 40 units.

Next Session Preview

Up: Area Problems & Applications

- Area under a single curve
- Area between a curve and the x-axis
- Handling regions where $f(x) < 0$
- Exponential and logarithmic integrals
- Business applications: total profit, accumulated production

. . .

Tip

Complete Tasks 06-02

- Practice evaluating definite integrals
- Focus on the FTC and its applications
- Work on signed vs. total area problems