Session 06-01 - Antiderivatives & Indefinite Integrals

Section 06: Integral Calculus

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Entry Quiz - 10 Minutes

Quick Review from Section 05

Test your understanding of Differential Calculus

1. Find $f'(x)$ for $f(x) = 3x^4 - 2x^2 + 5x - 7$
2. A profit function is $P(x) = -x^2 + 80x - 400$. Find the production level that maximizes profit.
3. For $g(x) = x^3 - 6x^2 + 9x + 1$, find all critical points and classify them.
4. If $C'(x) = 6x + 10$ represents marginal cost, what does $C'(5) = 40$ mean in business terms?

Homework Discussion - 15 Minutes

Your questions from Section 05

Focus on differential calculus applications

- Challenges with optimization problems
- Curve sketching and second derivative test
- Function determination (Funktionsscharen)
- Any remaining questions before we move forward

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> **ℹ Note**
>
> Today we begin Integral Calculus - the reverse of differentiation!

Learning Objectives

What You'll Master Today

- Understand the antiderivative concept as reversing differentiation
- Apply the power rule for integration: $\int x^n \, dx$

- Use sum and constant rules to integrate polynomials
- Interpret the constant of integration (+C) and families of functions
- Solve initial value problems to find specific antiderivatives
- Verify antiderivatives by differentiation
- Apply integration to business (marginal cost ⟶ total cost)

. . .

ℹ Note

Integration is the reverse of differentiation - today we learn to "undo" derivatives!

Part A: The Antiderivative Concept

From Derivatives to Antiderivatives

Remember differentiation?

$$f(x) = x^3 \quad \xrightarrow{\text{differentiate}} \quad f'(x) = 3x^2$$

. . .

Now we reverse the process:

$$F'(x) = 3x^2 \quad \xleftarrow{\text{antidifferentiate}} \quad F(x) = x^3$$

. . .

! Definition: Antiderivative (Stammfunktion)

$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

In German: F is the Stammfunktion of f.

The Question We're Solving

Given a derivative, find the original function

. . .

Example Questions:

- If $f'(x) = 2x$, what could $f(x)$ be?
- If the marginal cost is $C'(x) = 50$, what is the total cost function?
- If velocity is $v(t) = 3t^2$, what is the position function?

. . .

💡 Tip

Business Context: We often know the rate of change (marginal cost, growth rate) and need to find the total quantity (total cost, population).

Finding: First Examples

Question: If $F'(x) = 2x$, what is $F(x)$?

. . .

Think: What function, when differentiated, gives $2x$?

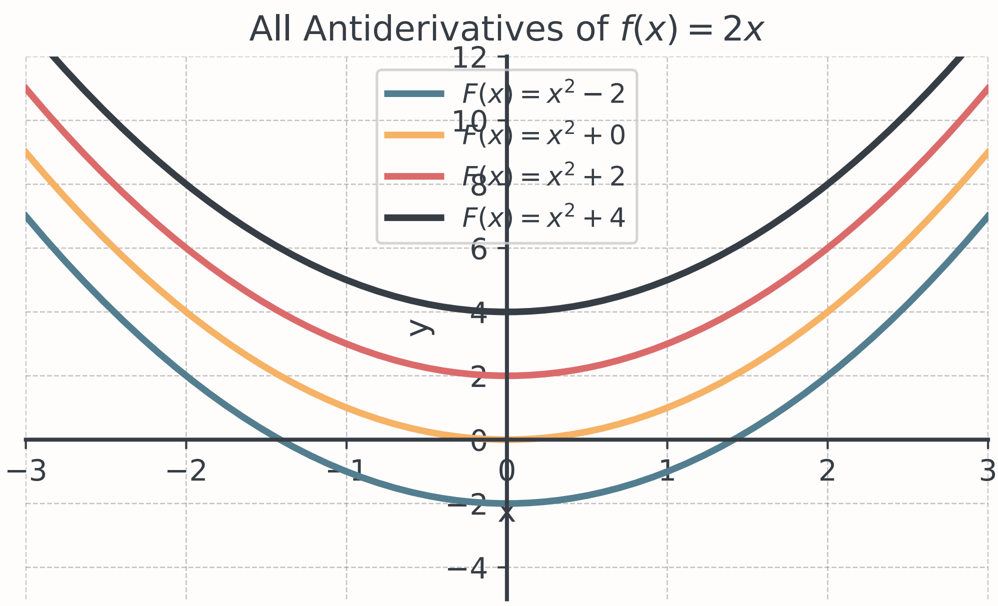
- If $F(x) = x^2$, then $F'(x) = 2x$ ✓
- But also: $F(x) = x^2 + 5$ gives $F'(x) = 2x$ ✓
- And: $F(x) = x^2 - 100$ gives $F'(x) = 2x$ ✓

. . .

⚠ Warning

Problem: There are infinitely many antiderivatives!

The Family of Antiderivatives



. . .

All these curves have the same slope at each x-value!

The Constant of Integration

Solution: We write the general antiderivative with a constant C:

...

$$\int 2x\,dx = x^2 + C$$

...

- The symbol $\int$ means “integrate” (find the antiderivative)
- dx indicates we integrate with respect to x
- $+C$ represents any constant (the constant of integration)
- This notation is called an indefinite integral

...

> ! Important
>
> Always include +C for indefinite integrals!

Part B: The Power Rule for Integration

Reversing the Power Rule

Recall the power rule for derivatives:

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

...

Reverse it for integration:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for} \quad n \neq -1$$

...

> Tip
>
> Memory trick: “Add 1 to the power, divide by the new power”

Power Rule Examples

Example 1: $\int x^3\,dx$

...

$$\int x^3\,dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$$

...

Check: $\frac{d}{dx}\left[\frac{x^4}{4} + C\right] = \frac{4x^3}{4} = x^3$ ✓

...

Example 2: $\int x^5 \, dx$

...

$$\int x^5 \, dx = \frac{x^6}{6} + C$$

Special Cases

Constant function: $\int 1 \, dx$ (or $\int dx$)

...

Think of $1 = x^0$:

$$\int x^0 \, dx = \frac{x^{0+1}}{0+1} + C = x + C$$

...

Linear function: $\int x \, dx$

$$\int x^1 \, dx = \frac{x^2}{2} + C$$

...

> **i Note**
>
> The antiderivative of a constant is a line, the antiderivative of x is a parabola.

Negative and Fractional Powers

Negative powers: $\int x^{-2} \, dx$

...

$$\int x^{-2} \, dx = \frac{x^{-2+1}}{-2+1} + C = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

...

Fractional powers: $\int x^{1/2} \, dx = \int \sqrt{x} \, dx$

...

$$\int x^{1/2} \, dx = \frac{x^{3/2}}{3/2} + C = \frac{2x^{3/2}}{3} + C = \frac{2\sqrt{x^3}}{3} + C$$

...

> ⚠ Warning
>
> The power rule fails when $n = -1$ (we would divide by zero). We'll handle $\int \frac{1}{x}\,dx$ later.

Part C: Sum and Constant Multiple Rules

Constant Multiple Rule

If k is a constant:

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$$

. . .

Example: $\int 5x^3\,dx$

. . .

$$\int 5x^3\,dx = 5\int x^3\,dx = 5 \cdot \frac{x^4}{4} + C = \frac{5x^4}{4} + C$$

. . .

> Tip
>
> Pull constants outside the integral, then integrate!

Sum and Difference Rules

For sums and differences:

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

. . .

Example: $\int (x^2 + 3x - 5)\,dx$

. . .

$$= \int x^2\,dx + \int 3x\,dx - \int 5\,dx$$

. . .

$$= \frac{x^3}{3} + 3 \cdot \frac{x^2}{2} - 5x + C = \frac{x^3}{3} + \frac{3x^2}{2} - 5x + C$$

Polynomial Integration

Example: $\int(4x^3 - 6x^2 + 2x - 7)\,dx$

- $\int 4x^3\,dx = 4 \cdot \frac{x^4}{4} = x^4$
- $\int 6x^2\,dx = 6 \cdot \frac{x^3}{3} = 2x^3$
- $\int 2x\,dx = 2 \cdot \frac{x^2}{2} = x^2$
- $\int 7\,dx = 7x$

. . .

Answer: $\int(4x^3 - 6x^2 + 2x - 7)\,dx = x^4 - 2x^3 + x^2 - 7x + C$

. . .

Check: $\frac{d}{dx}[x^4 - 2x^3 + x^2 - 7x + C] = 4x^3 - 6x^2 + 2x - 7$ ✓

Practice - 10 Minutes

Practice: Basic Integration

Work individually for 3 minutes

Find the following antiderivatives:

1. $\int 6x^2\,dx$
2. $\int(x^4 - 3x^2 + 1)\,dx$
3. $\int \frac{4}{x^3}\,dx$ (rewrite as power first)
4. $\int(2\sqrt{x} + 3)\,dx$

Integration Drill

Work in pairs

Evaluate these integrals and verify by differentiation:

1. $\int(3x^2 + 4x + 5)\,dx$
2. $\int(x^3 - 2x + 4)\,dx$
3. $\int 10x^4\,dx$
4. $\int(x^2 + \frac{1}{x^2})\,dx$

Break - 10 Minutes

Part D: Initial Value Problems

Finding a Specific Antiderivative

Problem: The general antiderivative has infinitely many solutions. How do we find a specific one?

. . .

Solution: Use an initial condition, a known point on the function.

- Given $f'(x)$ and a condition like $f(a) = b$
- Find the specific function $f(x)$.

. . .

Steps:

1. Find the general antiderivative with $+C$
2. Substitute the initial condition to solve for C
3. Write the specific solution

Example: Initial Value Problem

Problem: Find $f(x)$ if $f'(x) = 6x^2 - 4$ and $f(1) = 5$.

. . .

Step 1: Find the general antiderivative

. . .

$$f(x) = \int (6x^2 - 4)\, dx = 2x^3 - 4x + C$$

. . .

Step 2: Use the initial condition $f(1) = 5$

. . .

$$f(1) = 2(1)^3 - 4(1) + C = 5; C = 7$$

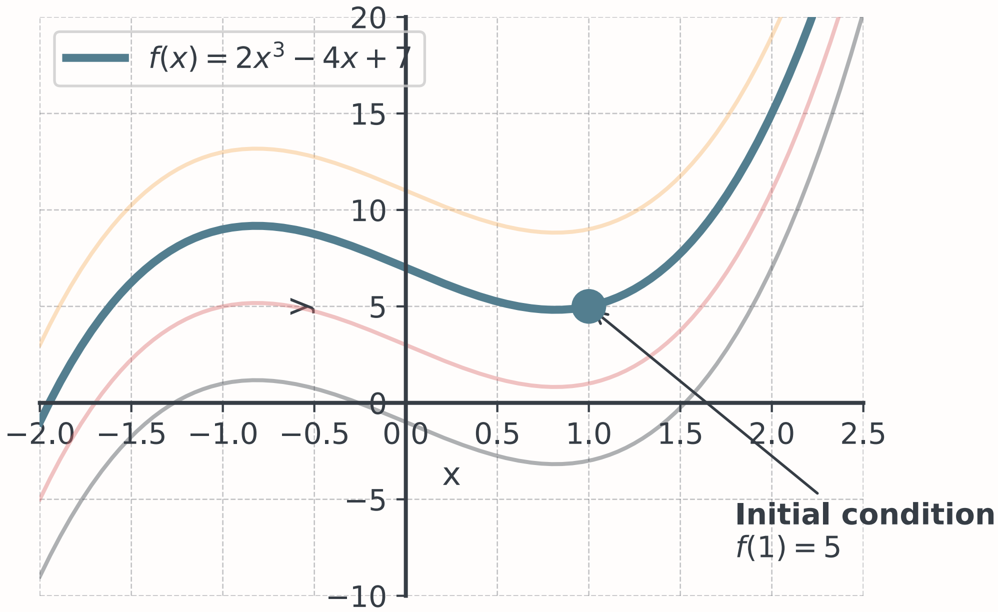
. . .

Step 3: Write the specific solution

. . .

$$f(x) = 2x^3 - 4x + 7$$

Visualizing the Initial Condition



> **! Important**
>
> The initial condition selects one curve from the family!

Total Cost from Marginal Cost

Scenario: A manufacturing company knows its marginal cost function:

$$MC(x) = C'(x) = 0.3x^2 - 2x + 50$$

The fixed costs are €1,000 (cost when producing 0 units).

. . .

Question: Find the total cost function $C(x)$.

. . .

Step 1: Integrate marginal cost

. . .

$$C(x) = \int (0.3x^2 - 2x + 50)\,dx = 0.1x^3 - x^2 + 50x + C$$

Completing the Business Problem

Step 2: Use the initial condition $C(0) = 1000$

. . .

$$C(0) = 0.1(0)^3 - (0)^2 + 50(0) + C = 1000$$

$$C = 1000$$

...

Step 3: Final answer

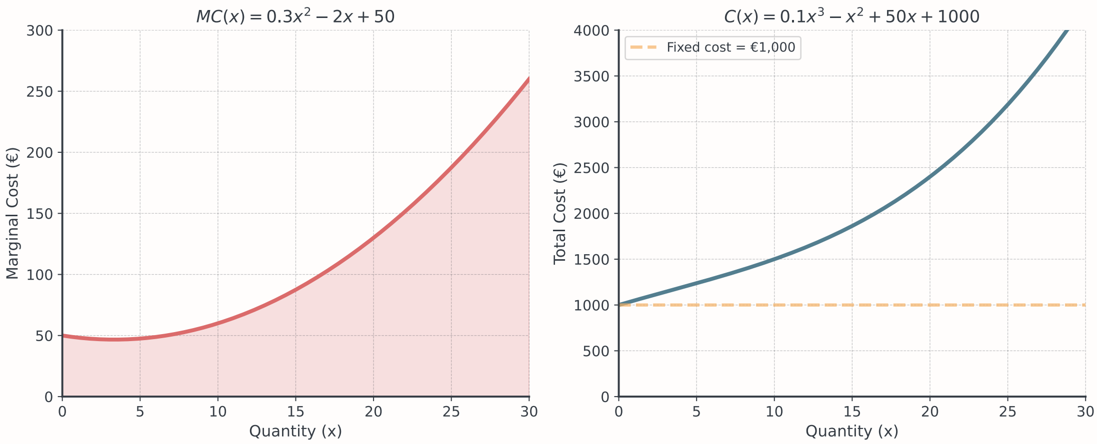
$$C(x) = 0.1x^3 - x^2 + 50x + 1000$$

...

Tip

The constant C represents the fixed costs, costs that don't depend on production level.

Visualizing Cost Functions



...

! Important

The marginal cost (derivative) shows the rate; the total cost (antiderivative) shows the accumulated total.

Part E: Velocity and Position

Application: Motion

Key relationships:

- Position: $s(t)$ = where the object is
- Velocity: $v(t) = s'(t)$ = rate of change of position
- Acceleration: $a(t) = v'(t) = s''(t)$ = rate of change of velocity

. . .

Reversing the relationships:

- If you know velocity, integrate to find position: $s(t) = \int v(t)\,dt$
- If you know acceleration, integrate to find velocity: $v(t) = \int a(t)\,dt$

Finding Position from Velocity

Problem: A car's velocity is $v(t) = 3t^2 + 2t$ m/s, where t is in seconds. At $t = 0$, the car is at $s = 10$ meters. Find the position function $s(t)$.

. . .

Step 1: Integrate velocity to get position

. . .

$$s(t) = \int (3t^2 + 2t)\,dt = t^3 + t^2 + C$$

. . .

Step 2: Use initial condition $s(0) = 10$

. . .

$$s(0) = 0 + 0 + C = 10 \implies C = 10$$

. . .

$$s(t) = t^3 + t^2 + 10$$

Verifying Our Answer

Check: Does $s'(t) = v(t)$?

. . .

$$s(t) = t^3 + t^2 + 10$$
$$s'(t) = 3t^2 + 2t = v(t) \quad \checkmark$$

. . .

Check: Does $s(0) = 10$?

. . .

$$s(0) = 0 + 0 + 10 = 10 \quad \checkmark$$

. . .

> ⚠ Warning
>
> Always check your antiderivative by differentiating to recover the original function and substituting the initial condition!

Guided Practice - 20 Minutes

Practice Set A: Basic Antiderivatives

Work individually for 5 minutes

Find the following indefinite integrals:

1. $\int (5x^4 - 3x^2 + 2)\, dx$
2. $\int (x^3 + x^{-1/2})\, dx$
3. $\int \frac{6}{x^4}\, dx$
4. $\int (4\sqrt{x} - \frac{3}{x^2})\, dx$

Practice Set B: Initial Value Problems

Work individually for 7 minutes

Solve these initial value problems:

1. $f'(x) = 4x^3 - 6x$, $f(2) = 10$. Find $f(x)$.
2. $g'(x) = 3x^2 + 4$, $g(0) = 5$. Find $g(x)$.
3. The marginal revenue is $MR(x) = 100 - 2x$. Revenue is €0 when no units are sold. Find $R(x)$.

Practice Set C: Business Applications

Work in pairs for 8 minutes

1. A company's marginal cost is $MC(x) = 20 + 0.4x$ euros per unit. Fixed costs are €500. Find the total cost function $C(x)$ and calculate the total cost of producing 50 units.
2. A product's marginal profit is $MP(x) = 80 - 4x$ euros per unit. The company breaks even (profit = 0) when selling 0 units. Find the profit function $P(x)$ and determine how many units maximize profit.

Coffee Break - 15 Minutes

Business Applications Deep Dive

From Rates to Totals

Integration connects rates of change to accumulated totals:

Rate Function (Derivative)	Total Function (Antiderivative)
Marginal cost $C'(x)$	Total cost $C(x)$
Marginal revenue $R'(x)$	Total revenue $R(x)$
Marginal profit $P'(x)$	Total profit $P(x)$
Production rate	Total production
Growth rate	Total quantity

. . .

> **i Note**
>
> The constant of integration often represents a fixed quantity (fixed costs, initial inventory, starting capital).

Example: Complete Cost Analysis

Scenario: A furniture manufacturer has:

- Marginal cost: $MC(x) = 0.02x^2 - 2x + 100$ euros per unit
- Fixed costs: €5,000
- Each unit sells for €150

. . .

Tasks:

1. Find the total cost function $C(x)$
2. Find the revenue function $R(x)$
3. Find the profit function $P(x)$
4. Determine the break-even points

Solution Part 1: Cost Function

Step 1: Integrate marginal cost

$$C(x) = \int (0.02x^2 - 2x + 100)\, dx$$

. . .

$$C(x) = \frac{0.02x^3}{3} - x^2 + 100x + C$$

. . .

Step 2: Apply initial condition $C(0) = 5000$

$$C = 5000$$

. . .

Answer: $C(x) = \frac{x^3}{150} - x^2 + 100x + 5000$

Solution Part 2: Revenue and Profit

Revenue: (Price × Quantity)

$$R(x) = 150x$$

...

Profit: (Revenue – Cost)

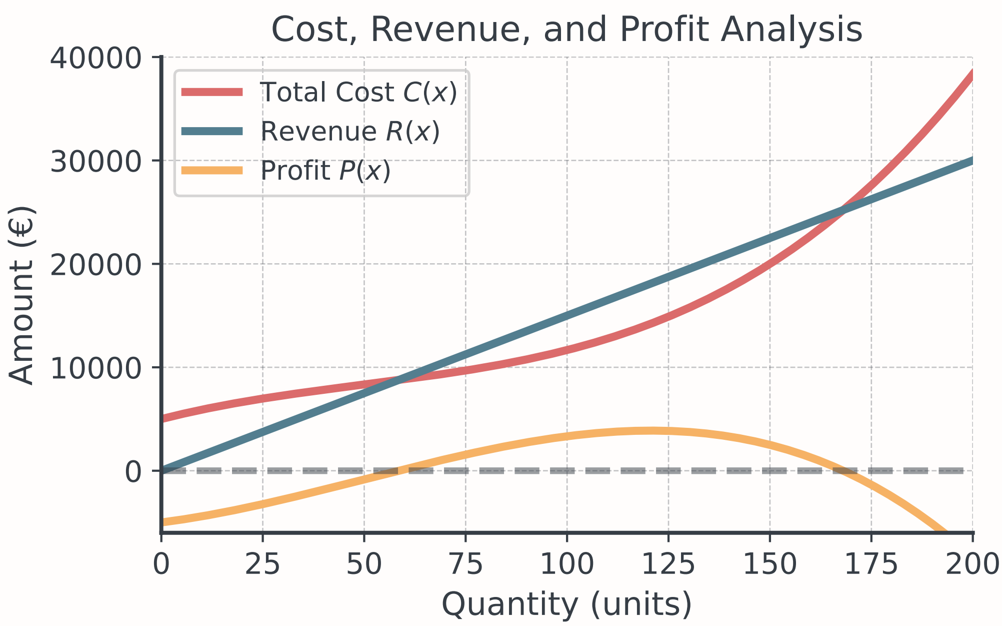
$$P(x) = R(x) - C(x)$$

$$P(x) = 150x - \left(\frac{x^3}{150} - x^2 + 100x + 5000\right)$$

...

$$P(x) = -\frac{x^3}{150} + x^2 + 50x - 5000$$

Visualizing the Business Model



Collaborative Problem-Solving

Group Challenge: Startup Analysis

Scenario: A tech startup is launching a new app. Their data analytics team has modeled:

Marginal cost (per user): $MC(x) = 0.01x + 5$ euros, where x is thousands of users

Fixed costs: €50,000 (servers, development, etc.)

Revenue per user: €12 per thousand users

Group Challenge: Tasks

Work in groups and then we compare

1. Find the total cost function $C(x)$ for x thousand users
2. Find the revenue function $R(x)$
3. Find the profit function $P(x)$
4. Calculate the profit/loss at 1,000 users, 5,000 users, and 10,000 users
5. Find the break-even point(s)
6. At what user count is profit maximized?

Wrap-Up & Key Takeaways

Today's Essential Concepts

- Antiderivative: If $F'(x) = f(x)$, then $F(x)$ is antiderivative of $f(x)$
- Power rule: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
- Sum/Constant rules: Split integrals, pull out constants
- +C is essential: Represents the family of all antiderivatives
- Initial conditions: Fix the value of C to find a specific function
- Verification: Always differentiate to check your answer

...

> ! Important
>
> Next session: We'll learn about definite integrals and the Fundamental Theorem of Calculus!

Integration Formulas Summary

Function	Antiderivative
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
k (constant)	$kx + C$
$kf(x)$	$k\int f(x)\,dx$
$f(x) \pm g(x)$	$\int f(x)\,dx \pm \int g(x)\,dx$

...

> Tip
>
> Integration is "undo-ing" differentiation. Ask yourself: "What function, when differentiated, gives me this?"

Final Assessment - 5 Minutes

Quick Check

Work individually, then compare

1. Find $\int (2x^3 - 5x + 3)\,dx$
2. Given $f'(x) = 6x - 2$ and $f(1) = 4$, find $f(x)$.
3. A company's marginal cost is $MC(x) = 30 + 0.5x$. Fixed costs are €200. What is the cost of producing 20 units?

Next Session Preview

Coming Up: Definite Integrals & FTC

- The definite integral as a signed area
- The Fundamental Theorem of Calculus
- Evaluating integrals using $F(b) - F(a)$
- Properties of definite integrals
- Applications to accumulated quantities

. . .

> Tip
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> Complete Tasks 06-01
>
> - Practice basic antiderivatives until they're automatic
> - Work through initial value problems
> - Focus on business applications (cost, revenue, profit)