Session 05-05 - Graphical Calculus Mastery

Section 05: Differential Calculus

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Entry Quiz - 10 Minutes

Quick Review from Session 05-04

Test your understanding of chain rule and implicit differentiation

1. Differentiate $f(x) = (2x^3 - 5)^4$ using the chain rule.
2. Find $\frac{dy}{dx}$ if $x^2 + xy = 10$.
3. If $f'(3) = 0$, what does this tell about the graph of $f(x)$ at $x = 3$?
4. If $f''(x) > 0$ for all x in an interval, what does this tell you about the shape of $f(x)$?

Homework Discussion - 15 Minutes

Your questions from Session 05-04

What questions do you have regarding the previous session?

Learning Objectives

What You'll Master Today

- Sketch $f'(x)$ from the graph of $f(x)$ by analyzing slopes
- Determine properties of $f(x)$ from the graph of $f'(x)$
- Understand concavity through second derivatives $f''(x)$
- Identify critical points and their classification
- Find inflection points where concavity changes
- Master visual analysis - a heavily tested exam skill!

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> **ℹ Key Insight**
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> Being able to translate between graphs of f, f', and f'' is one of the most important visual skills in calculus and frequently tested!

Part A: From Function to Derivative

Understanding the Derivative Graph

The graph of $f'(x)$ shows the slope of $f(x)$ at each point.

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What to look for in $\boldsymbol{f(x)}$:

- Where f is increasing $\longrightarrow f'(x) > 0$ (derivative is positive)
- Where f is decreasing $\longrightarrow f'(x) < 0$ (derivative is negative)
- Where f has a horizontal tangent $\longrightarrow f'(x) = 0$ (crosses x-axis)
- Where f is steep $\longrightarrow |f'(x)|$ is large
- Where f is flat $\longrightarrow |f'(x)|$ is small

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> **i Note**
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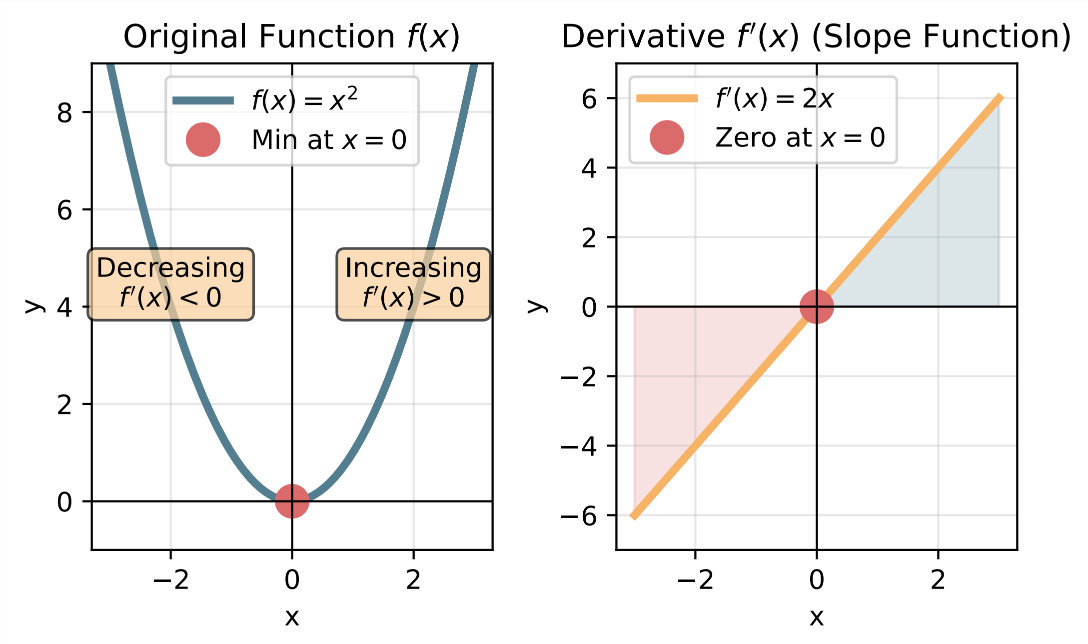
> This shouldn't be too complicated, right?

Sketching f' from f

A parabola and its derivative: $f(x) = x^2$



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> **Tip**
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> f has minimum at $x = 0 \longrightarrow f'$ crosses zero

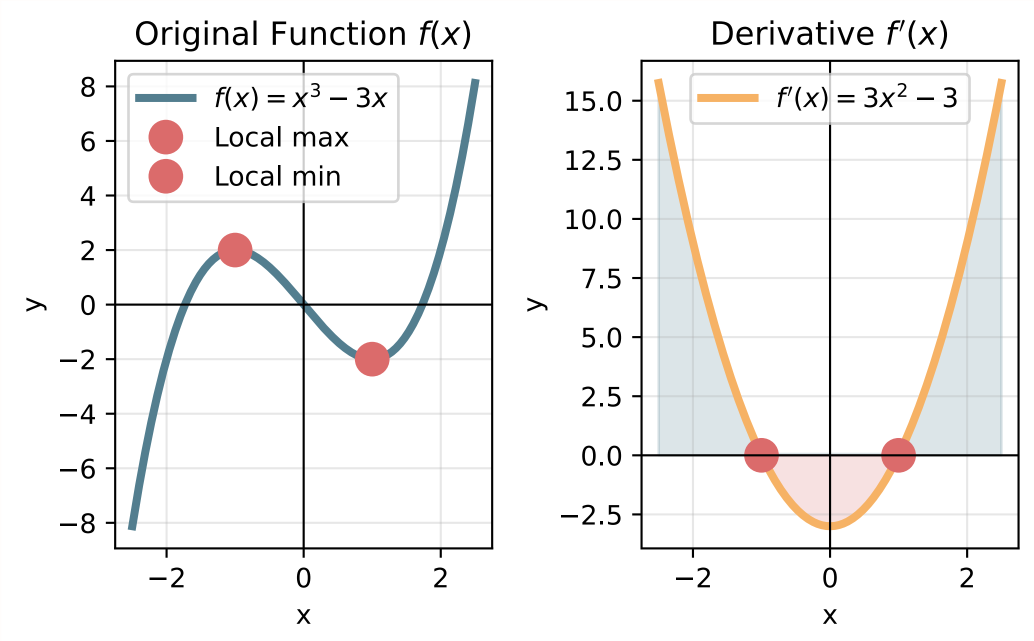
Step-by-Step: Sketching the Derivative

The process is rather straightforward:

1. Identify critical points: Where does f have horizontal tangents? Mark these as zeros of f'
2. Determine sign: Where is f increasing/decreasing? Make f' positive/negative accordingly
3. Consider steepness:
   - Where is f very steep? Make $|f'|$ large.
   - Where is f nearly flat? Make $|f'|$ small.
4. Check concavity: Is f' increasing or decreasing? This tells you about the concavity of f

Complex Example

A cubic function and its derivative.



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> 💡 Tip
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> f increasing, then decreasing, then increasing $\longrightarrow$ f' crosses zero twice

Critical Points

A point $x = c$ where either:

- $f'(c) = 0$ (horizontal tangent), or
- $f'(c)$ does not exist (corner, cusp, vertical tangent)

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> 💡 Classification of Critical Points
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> - Local maximum: f' changes from positive to negative
> - Local minimum: f' changes from negative to positive
> - Neither: f' doesn't change sign (e.g., $f(x) = x^3$ at $x = 0$)

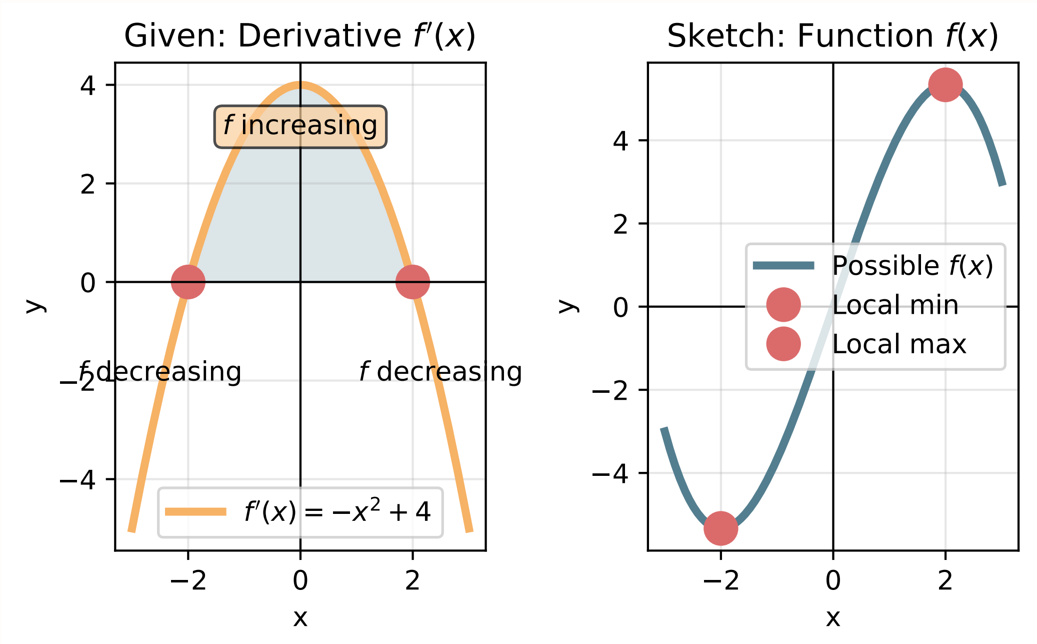
Part B: From Derivative to Function

Reading Information from $f'(x)$

Given the graph of $f'(x)$, we can determine:

1. Where f is increasing/decreasing:
   - $f'(x) > 0 \rightarrow f$ is increasing
   - $f'(x) < 0 \rightarrow f$ is decreasing
2. Where f has local extrema: f' crosses zero
   - Sign change determines type!
3. Where f is steepest: Where $|f'(x)|$ is largest
4. Relative heights: Cannot determine absolute y-values!
   - Can determine relative changes!

Example: Reading from $f'(x)$



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> Tip
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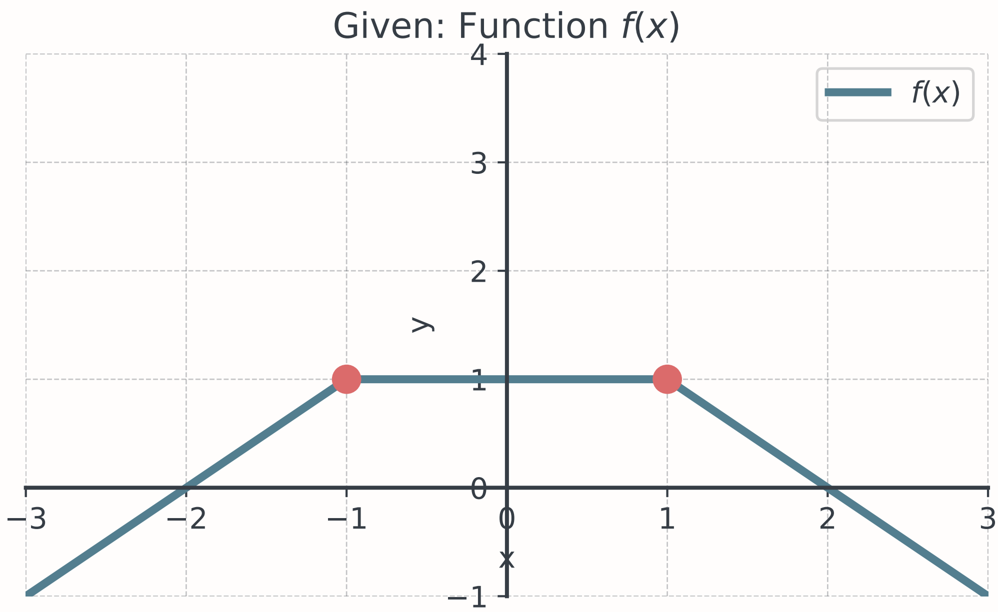
> - $f' > 0$ on $(-2, 2) \longrightarrow f$ increasing, $f' < 0$ for $|x| > 2 \longrightarrow f$ decreasing
> - f' changes at $x = -2, 2 \longrightarrow f$ has local extrema

Quick Practice - 10 Minutes

Individual Exercise I

Sketch the derivative of this function!



Individual Exercise II

The Questions:

a) Where is $f'(x) > 0$, $f'(x) < 0$, and $f'(x) = 0$?

b) Sketch the graph of $f'(x)$.

c) At what points does $f'(x)$ not exist?

Break - 10 Minutes

Part C: Second Derivatives and Concavity

Understanding Concavity

Describes the curving behavior of a function.

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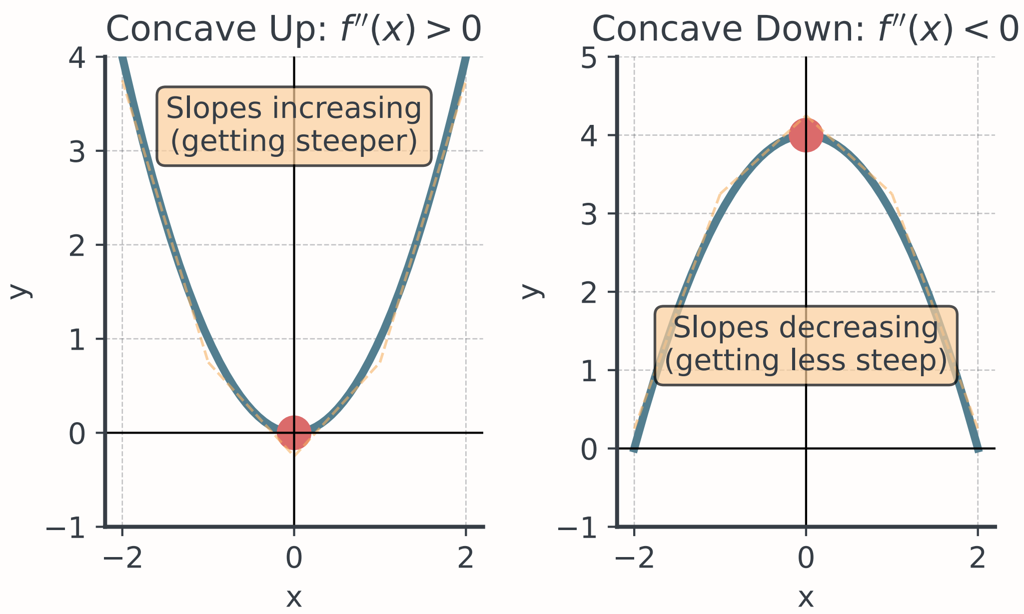
- Concave up (?): Curves upward like a smile
  - $f''(x) > 0$
  - The slope $f'(x)$ is increasing
  - "Holds water"
- Concave down (?): Curves downward like a frown
  - $f''(x) < 0$
  - The slope $f'(x)$ is decreasing
  - "Spills water"

Visualizing Concavity

Concave up vs. concave down



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i Note

$f''(x)$ tells us how $f'(x)$ is changing, which determines the shape of $f(x)$.

Inflection Points

A point where concavity changes!

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- $f''(x) = 0$ or $f''(x)$ does not exist
- Concavity changes (from ? to ? or vice versa)

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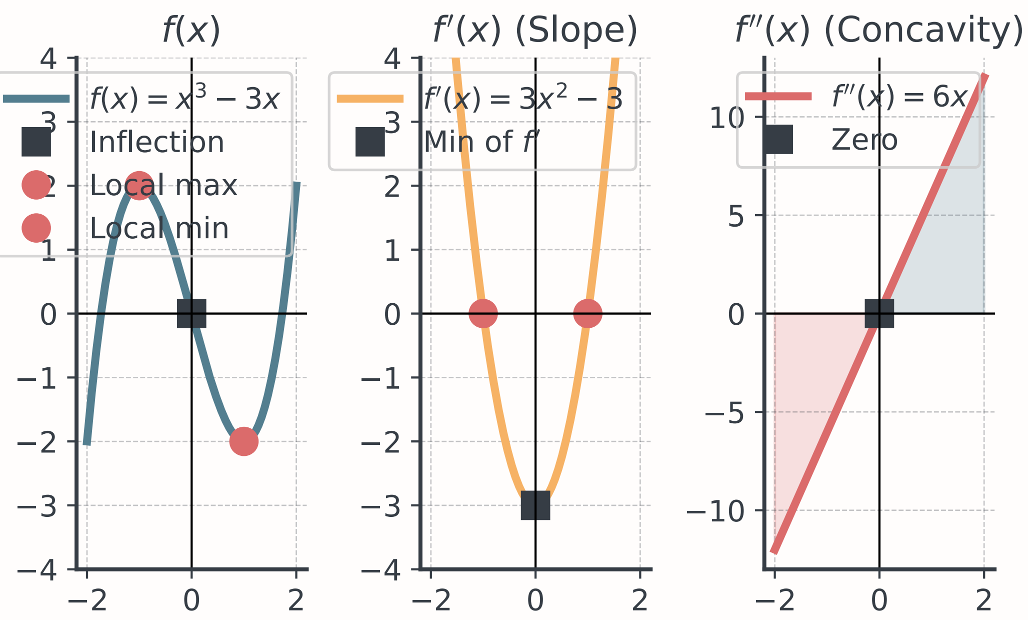
Example: $f(x) = x^3$

- $f'(x) = 3x^2$
- $f''(x) = 6x$
- $f''(0) = 0$ and concavity changes at $x = 0$
- So $(0, 0)$ is an inflection point

Relationships Between f, f', and f''



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> **! Important**
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> - Where $f'' < 0$: f concave down, f' decreasing, where $f'' > 0$: f concave up, f' increasing
> - Where $f'' = 0$: Inflection point in f, extremum in f'

Part D: Complete Analysis

The Complete Picture

Given a function $f(x)$, complete analysis involves:

1. Critical points: Solve $f'(x) = 0$
2. First derivative test: Check sign changes of f' to classify extrema
3. Inflection points: Solve $f''(x) = 0$ and check for concavity change
4. Intervals: Determine where f is increasing/decreasing and where concave up/down
5. Key points: Evaluate f at critical points and inflection points

Example: Complete Analysis

Analyze $f(x) = x^4 - 4x^3$

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First derivative:

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$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

...

Critical points: $x = 0, 3$

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Second derivative:

...

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

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Inflection points: $x = 0, 2$

Analysis Visualization

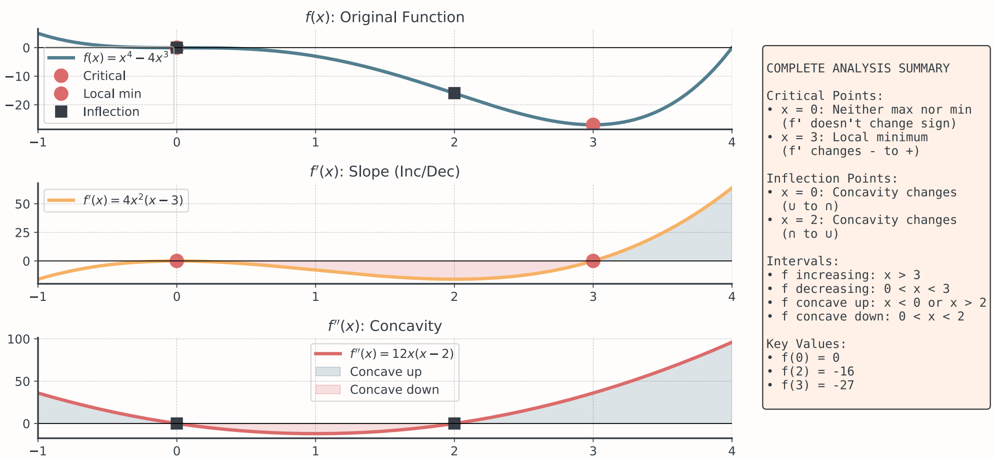


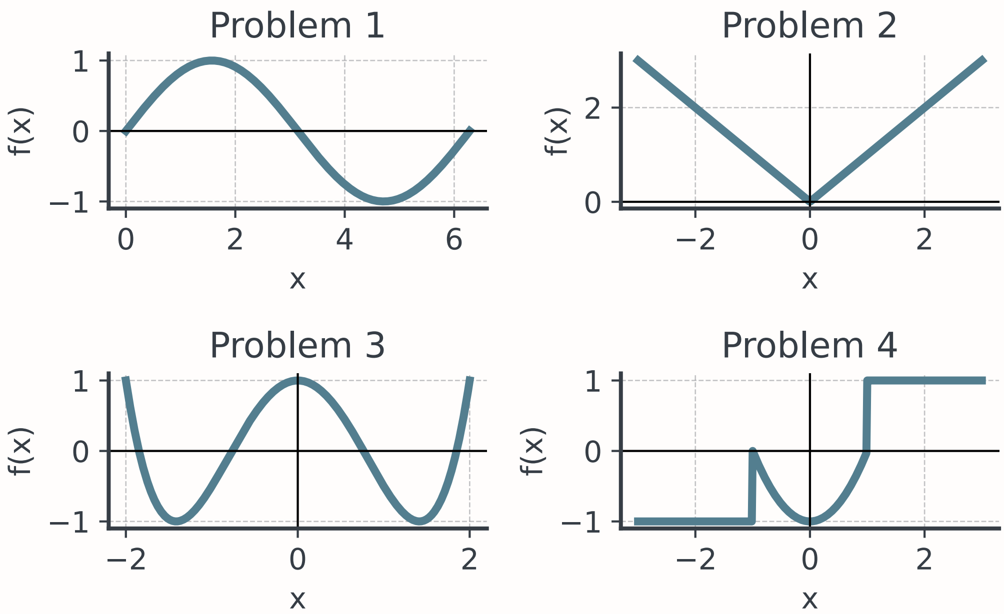
Figure 1: Complete analysis of $f(x) = x^4 - 4x^3$

Guided Practice

Practice Set A: Sketching Derivatives

For each function graphed below, sketch $f'(x)$ and identify:

- Where $f'(x) > 0, < 0, = 0$ and any points where f' does not exist



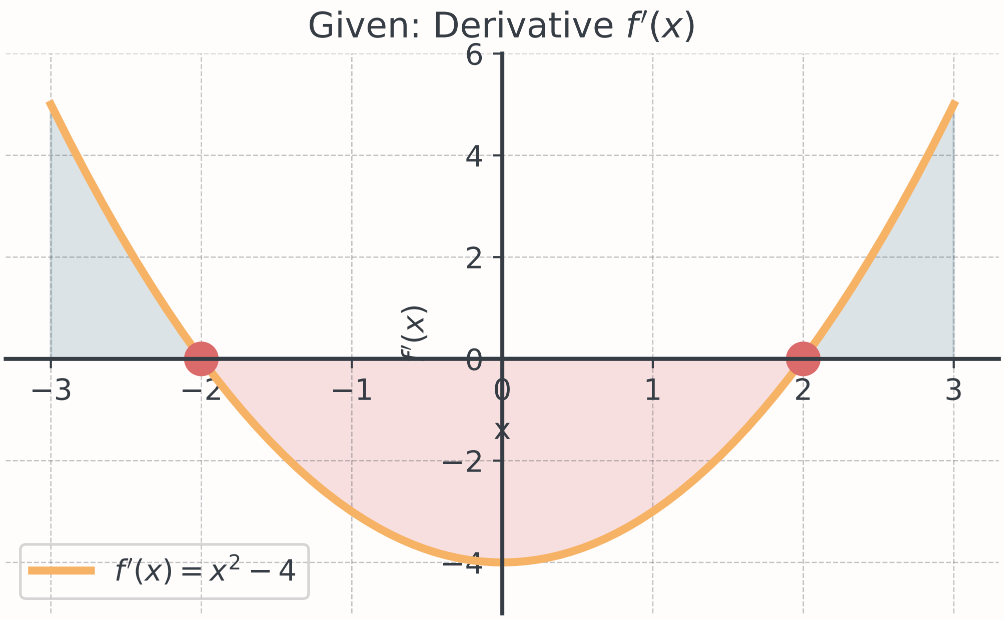
Practice B: From Derivative I

Continue working for 10 minutes

Given the graph of $f'(x)$, answer:

a) Where is $f(x)$ increasing/decreasing?
b) Where does $f(x)$ have local extrema? Classify them.
c) Sketch a possible graph of $f(x)$.

Practice B: From Derivative II



Quick Practice: Derivative Sketching

Work individually for 5 minutes

For each function described, sketch both $f(x)$ and $f'(x)$:

1. $f(x) = x^3 - 3x$ (cubic with local max and min)
2. $f(x) = |x - 2|$ (V-shape shifted right)
3. $f(x)$ is constant for $x < 0$, then increases linearly for $x \geq 0$
4. $f(x)$ has $f'(x) > 0$ everywhere but $f'(x) \to 0$ as $x \to \infty$

Coffee Break - 15 Minutes

Business Applications

Profit Function Analysis

Business Context: A company's monthly profit (in thousands €) is modeled by:

$$P(t) = -t^3 + 12t^2 - 36t + 50$$

where t is months since product launch.

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Questions:

1. When is profit increasing/decreasing?
2. When does profit reach local extrema?
3. When is the rate of profit change accelerating/decelerating?

Profit Analysis Solution

First derivative (profit rate of change):

$$P'(t) = -3t^2 + 24t - 36 = -3(t^2 - 8t + 12) = -3(t - 2)(t - 6)$$

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Critical points: $t = 2, 6$ months

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Second derivative (acceleration of profit change):

$$P''(t) = -6t + 24 = -6(t - 4)$$

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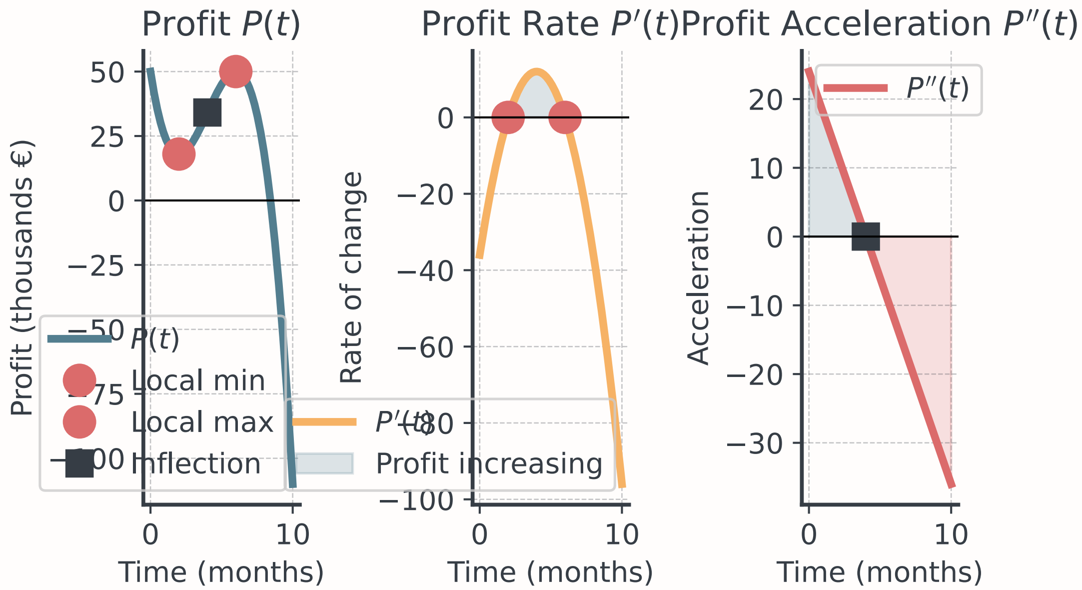
Inflection point: $t = 4$ months

Profit Visualization



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Question: How would you describe the behaviour?

Profit Interpretation

Reading Business Graphs: Quick Rules

- $P'(t) > 0 \rightarrow$ Profit is growing (good news!)
- $P'(t) < 0 \rightarrow$ Profit is shrinking (warning sign)
- $P'(t) = 0 \rightarrow$ Profit has reached a turning point (decision time)

The sign of the derivative tells you the direction of change!

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Inflection Points in Business

The inflection point ($P''(t) = 0$ at month 4) marks where:

- Before: Profit is accelerating (growth speeding up)
- After: Profit is decelerating (growth slowing down)

Business insight: Month 4 is when the company should start planning for the eventual peak. Growth is still positive but momentum is fading!

Collaborative Problem-Solving

Challenge: Complete Function Analysis

Scenario: An economist models consumer demand response using:

$$D(p) = \frac{100}{p+1} - 2$$

where D is demand (thousands of units) and p is price (€).

Tasks

Work in groups of 3-4

a) Find $D'(p)$ and interpret its sign. What does this tell you economically?

b) Find $D''(p)$. What does its sign tell you about how demand sensitivity changes with price?

c) Sketch the graphs of $D(p)$, $D'(p)$, and $D''(p)$ on $p \in [0, 10]$.

d) At what price is the rate of demand decrease (i.e., $|D'(p)|$) exactly 1 unit per €?

e) Business question: If you're a monopolist who can set price, explain using calculus concepts why you wouldn't set price arbitrarily high even though higher prices mean more revenue per unit.

Think-Pair-Share - 7 Minutes

Discussion Question

Think individually, then discuss with class

Question: Consider these scenarios:

- Company A: Stock price $S_A(t)$ is $S_A' > 0$ but $S_A'' < 0$
- Company B: Stock price $S_B(t)$ is $S_B' < 0$ but $S_B'' > 0$

1. Which company is in a better position right now?
2. Which company shows more promising momentum for the future?
3. In financial terms, what do S' and S'' represent?
4. Can you think of real-world examples of each scenario?

Wrap-Up & Key Takeaways

The Derivative Overview

If you know...	You can determine...
$f' > 0$	f is increasing
$f' < 0$	f is decreasing

If you know...	You can determine...
$f' = 0$	Possible local extremum
$f'' > 0$	f concave up, f' increasing
$f'' < 0$	f concave down, f' decreasing
$f'' = 0$	Possible inflection point

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> ⚠ Warning
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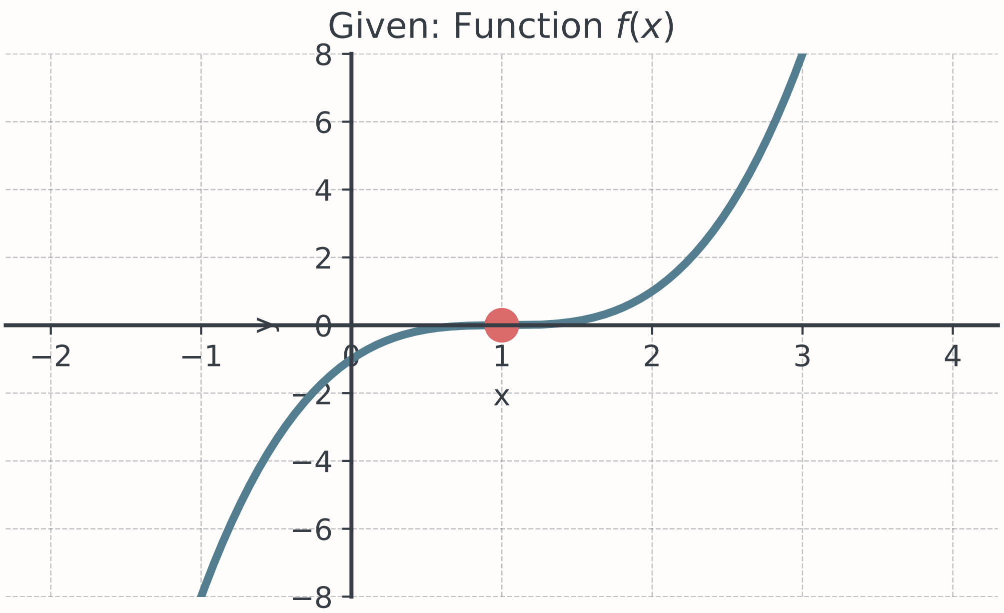
> These are heavily Tested Skills in FSP!!!

Final Assessment - 5 Minutes

Quick Check I

Consider this function:



Quick Check II

Work individually, then we compare

1. Where is $f'(x) > 0$?
2. At $x = 1$, is this a local max, local min, or neither?
3. Where is $f(x)$ concave up?
4. True or False: If $f'(c) = 0$, then f must have a local extremum at $x = c$.

Next Session Preview

Session 05-06

Optimization & Curve Sketching

- First and second derivative tests for extrema classification
- Global maxima/minima on closed intervals
- Complete curve sketching algorithm (6 steps)
- Business optimization: profit maximization, cost minimization
- Interpreting results in real-world context

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> **Tip**
>
> Complete Tasks 05-05!