Session 05-04 - Chain Rule & Implicit Differentiation

Section 05: Differential Calculus

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Entry Quiz - 10 Minutes

Quick Review from Session 05-03

Test your understanding of differentiation rules

1. Find the derivative of $f(x) = (x^2 + 1)(3x - 2)$.
2. Differentiate $g(x) = \frac{x^2}{x+1}$ using the quotient rule.
3. For $h(x) = 5x^{10} - 3x^2 + 7$, find $h'(x)$.
4. What is the tangent line to $f(x) = x^2$ at $x = 3$?

Homework Discussion - 15 Minutes

Your questions from Session 05-03

What questions do you have regarding the tasks?

Learning Objectives

What You'll Learn Today

- Master the chain rule for differentiating composite functions
- Combine chain rule with product and quotient rules effectively
- Use implicit differentiation when you can't solve for y
- Apply related rates to problems where quantities change over time
- Solve business problems with nested functions and changing rates

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> **i Key Insight**
>
> The chain rule unlocks differentiation of composite functions, one of the most powerful and widely used techniques in calculus!

Part A: The Chain Rule

The Challenge: Composite Functions

The Problem: How do we differentiate $(x^2 + 3x + 1)^{100}$?

- Could we expand it? No! Expansion would have hundreds of terms
- Current rules don't help: Not a simple power, product, or quotient
- This is a composite function: A function inside another function

. . .

The Structure:

- Outer function: $f(u) = u^{100}$
- Inner function: $u = x^2 + 3x + 1$
- Composite: $f(g(x)) = (x^2 + 3x + 1)^{100}$

The Chain Rule

Derivative of outer (evaluated at inner) times derivative of inner:

. . .

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

. . .

Or in Leibniz notation, if $\boldsymbol{y = f(u)}$ and $\boldsymbol{u = g(x)}$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

. . .

> ⚠ Warning
>
> Most Common Mistake: Forgetting to multiply by the derivative of the inner function!

Chain Rule Example

Example: Differentiate $f(x) = (x^2 + 3x + 1)^{100}$

. . .

Step 1: Identify inner and outer

- Outer: $f(u) = u^{100}$, so $f'(u) = 100u^{99}$
- Inner: $u = x^2 + 3x + 1$, so $u' = 2x + 3$

. . .

Step 2: Apply chain rule

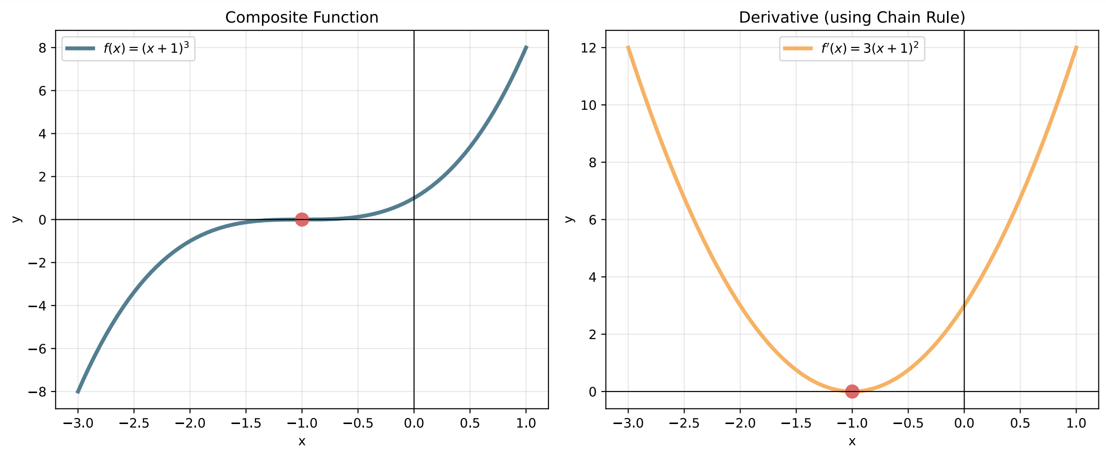
$$f'(x) = 100(x^2 + 3x + 1)^{99} \cdot (2x + 3)$$

...

Note

Notice: We never had to expand the original function!

Visualizing the Chain Rule



...

Note

- At $x = -1$, both the function and derivative equal zero
- The derivative is always non-negative (parabola opening upward)

More Chain Rule Examples

Example: Differentiate $g(x) = \sqrt{3x^2 + 5}$

...

Rewrite: $g(x) = (3x^2 + 5)^{1/2}$

- Outer: $u^{1/2}$, so derivative is $\frac{1}{2}u^{-1/2}$
- Inner: $3x^2 + 5$, so derivative is $6x$
- $g'(x) = \frac{1}{2}(3x^2 + 5)^{-1/2} \cdot 6x = \frac{6x}{2\sqrt{3x^2+5}} = \frac{3x}{\sqrt{3x^2+5}}$

...

Example: Differentiate $h(x) = \frac{1}{(2x-1)^3}$

...

Rewrite: $h(x) = (2x - 1)^{-3}$

...

- $h'(x) = -3(2x-1)^{-4} \cdot 2 = \frac{-6}{(2x-1)^4}$

Part B: Combining the Chain Rule

Chain Rule with Product Rule I

Strategy: When you have a product with composite functions:

1. Apply product rule first (outer operation)
2. Use chain rule for each composite factor

...

Example: Differentiate $f(x) = x^2(3x+1)^5$

Chain Rule with Product Rule II

Product rule:

- $u = x^2$, so $u' = 2x$
- $v = (3x+1)^5$, so $v' = 5(3x+1)^4 \cdot 3 = 15(3x+1)^4$ (chain rule!)
- $f'(x) = 2x(3x+1)^5 + x^2 \cdot 15(3x+1)^4$

...

Factor:

- $= (3x+1)^4[2x(3x+1) + 15x^2]$
- $= (3x+1)^4[6x^2 + 2x + 15x^2]$
- $= (3x+1)^4(21x^2 + 2x)$

Simplified Chain Applications

Example: Differentiate $h(x) = \sqrt{(2x+1)^3}$

...

Simplify first!

...

$$h(x) = (2x+1)^{3/2}$$

...

$$h'(x) = \frac{3}{2}(2x+1)^{1/2} \cdot 2$$

...

$$h'(x) = 3\sqrt{2x+1}$$

Quick Practice - 10 Minutes

Individual Exercise

Work individually for 10 minutes

Differentiate using the chain rule:

a) $f(x) = (5x + 2)^4$

b) $g(x) = \sqrt{x^2 - 1}$

c) $h(x) = (x^2 + 1)^3 (2x - 1)^2$

d) $k(x) = \frac{x}{(3x+1)^2}$

Break - 10 Minutes

Part C: Implicit Differentiation

Two Ways to Find a Derivative

Problem: Revenue constraint $pq = 10000$. Find $\frac{dq}{dp}$.

. . .

Solve First

1. $q = \frac{10000}{p} = 10000p^{-1}$ (Solve for q then differentiate)
2. $\frac{dq}{dp} = -10000p^{-2} = -\frac{10000}{p^2}$

Differentiate Directly

1. $\frac{d}{dp}[pq] = \frac{d}{dp}[10000]$ (Differentiate both sides with respect to p)
2. $\frac{d}{dp}[p] \cdot q + p \cdot \frac{d}{dp}[q] = 0$ (Apply product rule on the left)
3. $q + p\frac{dq}{dp} = 0$ (Solve for $\frac{dq}{dp}$)
4. $\frac{dq}{dp} = -\frac{q}{p}$ (Substitute the original constraint)
5. $\frac{dq}{dp} = -\frac{10000/p}{p} = -\frac{10000}{p^2}$

. . .

> Tip
>
> Both methods give $\frac{dq}{dp} = -\frac{10000}{p^2}$. Method 2 is called implicit differentiation.

Why Learn Implicit Differentiation?

Because sometimes Method 1 is impossible or complex!

. . .

Factory produces constant output $Q = 100$ with labor L and capital K:

$$L^{0.6} \cdot K^{0.4} = 100$$

...

When we write $L^{0.6} \cdot K^{0.4} = 100$, we're really saying:

$$L^{0.6} \cdot [K(L)]^{0.4} = 100$$

...

Question: What does this mean?

...

> Tip
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> Every variable is secretly a function!

The Main Idea

K depends on L, we just don't know the explicit formula!

$$L^{0.6} \cdot K^{0.4} = 100$$

...

The Chain Rule applies:

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$$\frac{d}{dL}[K^{0.4}] = 0.4 \cdot K^{-0.6} \cdot \frac{dK}{dL}$$

...

> Core Principle
>
> Whenever you differentiate a term containing K, multiply by $\frac{dK}{dL}$ because K is a function of L.

Production Example

Step 1: Differentiate both sides with respect to L

$$\frac{d}{dL}[L^{0.6} \cdot K^{0.4}] = \frac{d}{dL}[100]$$

...

Step 2: Apply product rule on left side and solve for $\frac{dK}{dL}$

$$0.6L^{-0.4} \cdot K^{0.4} + L^{0.6} \cdot 0.4K^{-0.6} \cdot \frac{dK}{dL} = 0$$

...

$$\frac{dK}{dL} = -\frac{0.6L^{-0.4} \cdot K^{0.4}}{0.4L^{0.6} \cdot K^{-0.6}} = -\frac{0.6}{0.4} \cdot \frac{K}{L} = -\frac{3K}{2L}$$

Interpreting the Result

What does this mean?

$$\frac{dK}{dL} = -\frac{3K}{2L}$$

. . .

- The negative sign: more labor $\rightarrow$ less capital needed (trade-off)
- The ratio $\frac{K}{L}$ matters: if $K = 20$ and $L = 10$, then $\frac{dK}{dL} = -3$
- Each additional unit of labor saves 3 units of capital
- Marginal Rate of Technical Substitution (MRTS) in economics!

. . .

> Tip
>
> The MRTS tells managers how to substitute between inputs while maintaining the same output level, crucial for cost minimization!

Example: Constant Revenue

A company sells a product with a constant revenue of 5000:

. . .

Practice: $pq = 5000$. Find $\frac{dq}{dp}$ using implicit differentiation.

. . .

Result: $\frac{dq}{dp} = -\frac{q}{p}$

. . .

Interpretation: At any point on the demand curve, a 1% price increase requires approximately a 1% quantity decrease to maintain revenue.

. . .

> When to Use Each Method
>
> - Solve first if it's easy (linear equations, simple fractions)
> - Implicit differentiation when solving is messy or impossible
> - You need the rate of change without fully solving the equation

Part D: Related Rates

Introduction to Related Rates

The Concept: When two or more quantities are related by an equation and both change with time, their rates of change are also related.

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General Process:

1. Write an equation relating the quantities
2. Differentiate both sides with respect to time t
3. Substitute known values
4. Solve for the unknown rate

...

> Tip
>
> Related rates problems use implicit differentiation with respect to time!

Growing Customer Base

Scenario: A company's revenue R depends on its customer base C:

$$R = 50\sqrt{C}$$

The company is gaining 100 new customers per month. How fast is revenue growing when $C = 10000$ customers?

...

Step 1: Given relationship $R = 50C^{1/2}$

...

Step 2: Differentiate to time $\frac{dR}{dt} = 50 \cdot \frac{1}{2}C^{-1/2} \cdot \frac{dC}{dt} = \frac{25}{\sqrt{C}} \cdot \frac{dC}{dt}$

...

Step 3: Substitute given: $\frac{dC}{dt} = 100$ customers/month, $C = 10000$

...

$$\frac{dR}{dt} = \frac{25}{\sqrt{10000}} \cdot 100 = \frac{25}{100} \cdot 100 = 25 \quad €/\text{month}$$

Business Related Rates Example

Scenario: A company's profit P and market share m are related by:

$$P = 1000m - 20m^2$$

where P is in k € and m is market share percentage. Market share is increasing at 2% per month. How fast is profit changing when $m = 15\%$?

...

Step1: Differentiate $\frac{dP}{dt} = 1000\frac{dm}{dt} - 40m\frac{dm}{dt} = (1000 - 40m)\frac{dm}{dt}$

...

Step 2: Substitute given $\frac{dm}{dt} = 2$ %/month, $m = 15$ %

...

$$\frac{dP}{dt} = (1000 - 40 \cdot 15)(2) = (1000 - 600)(2) = 800$$

...

Answer: Profit is increasing at €800,000 per month.

Guided Practice

Practice Set A: Chain Rule Applications

Work in pairs for 15 minutes

Differentiate the following:

a) $f(x) = \left(x^3 - 2x + 1\right)^{10}$
b) $g(x) = \sqrt{5x^2 + 3x - 1}$
c) $h(x) = \frac{1}{\left(x^2+1\right)^3}$
d) $k(x) = x(2x - 3)^4$

Practice Set B: Implicit & Related

Continue working in pairs for 10 minutes

Implicit Differentiation:

a) A company's marketing M and sales S satisfy $MS = 5000$. Find $\frac{dS}{dM}$.
b) Budget constraint: $2L + 3K = 120$ where L = labor, K = capital. Find $\frac{dK}{dL}$.

Related Rates:

c) Revenue $R = 100Q - 0.5Q^2$ and quantity grows at 10 units/month. Find $\frac{dR}{dt}$ when $Q = 50$.

Coffee Break - 15 Minutes

Business Applications

Nested Economic Functions

Business Context: A company's revenue depends on price p, which itself depends on quantity x:

- Price function: $p(x) = 100 - 0.5x$
- Revenue from price: $R(p) = p(200 - 2p)$

. . .

Question: Find $\frac{dR}{dx}$, rate of change of revenue with respect to quantity.

. . .

> **Note**
>
> Any idea how to approach this problem?

Chain Rule for Nested Functions I

. . .

Step 1: Express R in terms of p with $R(p) = 200p - 2p^2$

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$$\frac{dR}{dp} = 200 - 4p$$

. . .

Step 2: Find $\frac{dp}{dx}$ with $p(x) = 100 - 0.5x$

. . .

$$\frac{dp}{dx} = -0.5$$

. . .

Step 3: Apply chain rule $\frac{dR}{dx} = \frac{dR}{dp} \cdot \frac{dp}{dx}$

. . .

$$\frac{dR}{dx} = (200 - 4p)(-0.5)$$

Chain Rule for Nested Functions II

Now just substitute $p = 100 - 0.5x$ into expression for $\frac{dR}{dx}$:

…

$$\frac{dR}{dx} = (200 - 4(100 - 0.5x))(-0.5)$$

…

$$\frac{dR}{dx} = 100 - x$$

…

> ℹ Note
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> - At x, revenue changes at $(100 - x)$ € per unit
> - When $x = 50$: $\frac{dR}{dx} = 50$ (revenue still increasing)
> - When $x = 100$: $\frac{dR}{dx} = 0$ (revenue maximized!)

Collaborative Problem-Solving

Challenge: Expanding Retail Chain

A retail chain's profit P (in €1000) and stores n are related by:

$$P = 200\sqrt{n} - 5n$$

The company opens 4 new stores per year.

a) Write the relationship between $\frac{dP}{dt}$ and $\frac{dn}{dt}$.

b) How fast is profit changing when $n = 25$ stores?

c) How fast is profit changing when $n = 100$ stores?

d) At how many stores does profit stop growing? What does this mean for the company?

e) Discussion: Why does profit growth slow as the chain expands?

Think-Pair-Share

Discussion Question

Think individually, then discuss with class

Question: You're analyzing two companies:

- Company A: Revenue $R_A = (1000 + 50t)^{1.1}$ where t is years
- Company B: Revenue $R_B = 1100 + 50t$

At $t = 0$, both have revenue of 1000 (thousand €).

1. Which company's revenue is growing faster at $t = 0$?
2. Will this always be the case?

3. What does the exponent 1.1 tell you about A's growth strategy?

Wrap-Up & Key Takeaways

Today's Essential Techniques

Technique	When to Use	Key Idea
Chain Rule	Composite $f(g(x))$	Outer derivative × inner derivative
Chain + Product	Product with composites	Product rule first, then chain
Implicit Diff	Variables intertwined	Differentiate both, solve for derivative
Related Rates	Quantities change over time	Differentiate with respect to t

. . .

> ! Important
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> Never forget the inner derivative! This is the main mistake in chain rule problems.

Final Assessment - 5 Minutes

Quick Check

Work individually and then we compare

1. Differentiate $(3x^2+1)^4$.
2. Price p and quantity q satisfy $pq = 800$. Find $\frac{dq}{dp}$.
3. Profit $P = 50\sqrt{Q}$ and production grows at 8 units/month. How fast is profit growing when $Q = 100$?
4. True or False: $\frac{d}{dx}[f(g(x))] = f'(x) \cdot g'(x)$

Next Session Preview

Session 05-05

Graphical Calculus Mastery

- From f to f': Sketching derivatives from function graphs
- From f' to f: Determining function properties from derivative
- Second derivatives: Understanding f'' and concavity
- Critical points: Where $f'(x) = 0$ or doesn't exist

- Inflection points: Where $f''(x) = 0$ and concavity changes
- Heavily tested on exams! Visual analysis is crucial

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> Tip
>
> Complete Tasks 05-04!