Session 05-03 - Differentiation Rules & Tangent Lines

Section 05: Differential Calculus

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Entry Quiz - 10 Minutes

Quick Review from Session 05-02

Test your understanding of derivatives as rate of change

- 1. Using the limit definition, find f'(x) if $f(x) = 3x^2$.
- 2. What is the derivative of g(x) = 5x using the limit definition?
- 3. For the cost function C(x) = 100 + 20x, what is C'(x) and what does it mean in business terms?
- 4. If the slope of the tangent line to f(x) at x = 2 is 5, what is f'(2)?

Homework Discussion - 15 Minutes

Your questions from Session 05-02

What questions do you have regarding the tasks?

Learning Objectives

What You'll Master Today

- Master differentiation rules to avoid tedious limit calculations
- · Apply product and quotient rules correctly and efficiently
- Find equations of tangent lines at specific points on curves
- Use linear approximation for estimation in business contexts
- Choose optimal strategies for differentiating complex functions

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i Key Insight

Today we develop shortcuts that make differentiation fast and efficient, no more slow limit algebra!

Part A: Power, Constant, and Sum Rules

Why We Need Rules

The Problem: Computing derivatives from limit definition is tedious!

- For $f(x) = x^{10}$, the binomial expansion has 11 terms
- For $f(x) = (x^2 + 3x + 1)^{50}$, it's essentially impossible
- We need shortcuts that give the same answer faster

The Solution: Differentiation rules, formulas that work every time!

The Power Rule

Power Rule:

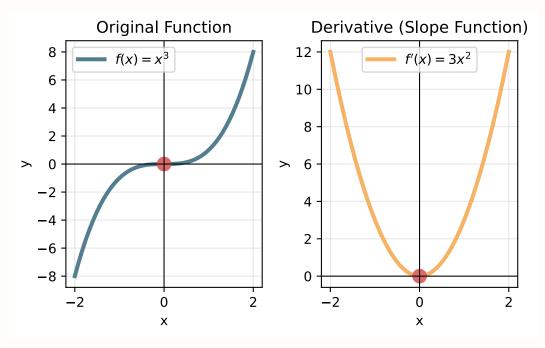
$$\frac{d}{dx}[x^n] = nx^{n-1}$$

where n is any real number.

Examples:

- $\begin{array}{l} \bullet \ \, \frac{d}{dx} \big[x^5 \big] = 5 x^4 \\ \bullet \ \, \frac{d}{dx} \big[\sqrt{x} \big] = \frac{d}{dx} \big[x^{1/2} \big] = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \\ \bullet \ \, \frac{d}{dx} \left[\frac{1}{x^3} \right] = \frac{d}{dx} \big[x^{-3} \big] = -3 x^{-4} = -\frac{3}{x^4} \end{array}$

Visualizing the Power Rule



• Where $f(x) = x^3$ is increasing, $f'(x) = 3x^2 > 0$

• At x = 0, f has a horizontal tangent, so f'(0) = 0

Constant Rules

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

where c is any constant.

• Constants don't change, so their rate of change is zero!

• Examples: $\frac{d}{dx}[7] = 0$, $\frac{d}{dx}[\pi] = 0$, $\frac{d}{dx}[-42] = 0$



This is one is very easy, right?

Constant Multiple Rules

Constant Multiple Rule:

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

- Pull constants out front!
- $\begin{array}{l} \bullet \ \ {\rm Examples:} \ \frac{d}{dx}[5x^3] = 5(3x^2) = 15x^2 \\ \bullet \ \frac{d}{dx}\big[\frac{3}{x^2}\big] = 3 \cdot \frac{d}{dx}\big[x^{-2}\big] = 3(-2x^{-3}) = -\frac{6}{x^3} \\ \end{array}$

Ţip

Do you get the idea? Essentially, we just keep the constant if multiplied with a variable.

The Sum Rule

Sum Rule:

$$\frac{d}{dx}[f(x)+g(x)]=f'(x)+g'(x)$$

Also works for differences: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Differentiate term by term!

•
$$\frac{d}{dx}[x^3 + x^2] = 3x^2 + 2x$$

• $\frac{d}{dx}[4x^5 - 2x^3 + 7] = 20x^4 - 6x^2 + 0 = 20x^4 - 6x^2$

•
$$\frac{d}{dx}\left[\sqrt{x} + \frac{1}{x}\right] = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$$

Combining the Basic Rules

Example: Differentiate $f(x) = 3x^4 - 2x^2 + 5x - 7$

. . .

$$f'(x) = 3(4x^3) - 2(2x) + 5(1) - 0 = 12x^3 - 4x + 5$$

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Example: Differentiate $g(x) = \frac{2}{x^3} + 4\sqrt{x} - 3$

. . .

$$g(x) = 2x^{-3} + 4x^{1/2} - 3$$

. . .

$$g'(x) = 2(-3x^{-4}) + 4\left(\frac{1}{2}x^{-1/2}\right) - 0 = -\frac{6}{x^4} + \frac{2}{\sqrt{x}}$$

Part B: The Product Rule

The Product Rule

Product Rule:

$$\frac{d}{dx}[u(x)\cdot v(x)] = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

"First times derivative of second, plus second times derivative of first"

. . .

▲ Common Product Rule Mistakes

- 1. Wrong: $(fg)' = f' \cdot g'$ (The product of derivatives)
- 2. Wrong: Forgetting one of the two terms
- 3. Right: (fg)' = f'g + fg' (Both terms, both factors in each)

If in doubt, check by expanding first when possible!

Product Rule Example

Example: Differentiate $f(x) = x^3(2x + 5)$

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Let
$$u = x^3$$
 and $v = 2x + 5$

•
$$u' = 3x^2$$

- v' = 2
- $f'(x) = (3x^2)(2x+5) + (x^3)(2)$
- $f'(x) = 6x^3 + 15x^2 + 2x^3 = 8x^3 + 15x^2$

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Ocheck by expanding first

• $f(x) = 2x^4 + 5x^3 \Rightarrow f'(x) = 8x^3 + 15x^3 = 8x^3 + 15x^2$

When to Use Product Rule?

Some rules of thumb to remember:

- Use it when: Expanding would be tedious: $(x^2 + 1)^{10}(x^3 2)^5$
- Skip it when: Easy to expand: $x(x+1) = x^2 + x$
- Remember: One factor is constant? Use constant multiple rule instead!

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i Note

Let's practice these rules!

Quick Practice - 10 Minutes

Individual Exercise

Work individually for 10 minutes

Differentiate the following:

a)
$$f(x) = x^7 - 3x^4 + 2x - 5$$

b)
$$g(x) = 4\sqrt{x} + \frac{2}{x} - 3$$

c)
$$h(x) = (x^2 + 1)(3x - 2)$$

d)
$$k(x) = x^4(x^2 - 5x + 1)$$

Break - 10 Minutes

Part C: The Quotient Rule

Differentiating Quotients

The Quotient Rule:

$$\frac{d}{dx} \left[\frac{u(x)}{v(x)} \right] = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$

"Low d-High minus High d-Low, over Low squared"

▲ Common Mistake

 $\left(\frac{f}{q}\right)' \neq \frac{f'}{q'}$ Always use the full quotient rule formula!

Quotient Rule Example

Example: Differentiate $f(x) = \frac{x^2+1}{x-3}$

Let $u = x^2 + 1$ and v = x - 3

- u' = 2x
- $f'(x) = \frac{(2x)(x-3)-(x^2+1)(1)}{(x-3)^2}$ $= \frac{2x^2-6x-x^2-1}{(x-3)^2}$ $= \frac{x^2-6x-1}{(x-3)^2}$

More Examples

Example: Differentiate $g(x) = \frac{2x}{x+1}$

$$g'(x) = \frac{2(x+1) - 2x(1)}{(x+1)^2} = \frac{2x + 2 - 2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Example: Differentiate $h(x) = \frac{x^2 - 4}{x^2 + 4}$

$$h'(x) = \frac{2x(x^2+4) - (x^2-4)(2x)}{(x^2+4)^2}$$

$$= \frac{2x^3 + 8x - 2x^3 + 8x}{(x^2 + 4)^2} = \frac{16x}{(x^2 + 4)^2}$$

Part D: Tangent Lines & Linear Approximation

Finding Tangent Lines (TL)

TL to y = f(x) at x = a has slope f'(a) and passes (a, f(a)).

Equation of Tangent Line:

$$y - f(a) = f'(a)(x - a)$$

or in slope-intercept form:

$$y = f(a) + f'(a)(x - a)$$

. . .

- 1. Find the point: Compute f(a)
- 2. Find the slope: Compute f'(a)
- 3. Write the equation using point-slope form

Tangent Line Example

Example: Find the tangent line to $f(x) = x^2 - 3x + 1$ at x = 2.

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Step 1: Find the point

•
$$f(2) = 4 - 6 + 1 = -1$$
, Point: $(2, -1)$

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Step 2: Find the slope

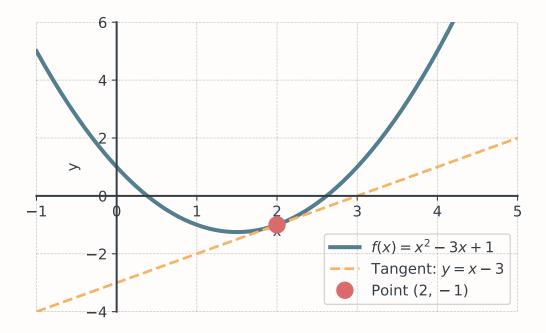
•
$$f'(x) = 2x - 3$$
, $f'(2) = 4 - 3 = 1$

. . .

Step 3: Equation

•
$$y - (-1) = 1(x - 2)$$
, $y = x - 3$

Visualizing the Tangent Line



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♀ Tip

Not too complicated, right?

Linear Approximation

Linear Approximation: For x near a:

$$f(x) \approx f(a) + f'(a)(x - a)$$

This uses the tangent line to estimate function values!

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Susiness Application

When x changes slightly from a, the change in f(x) is approximately:

$$\Delta f \approx f'(a) \cdot \Delta x$$

This is the basis of sensitivity analysis in economics!

Approximation Example

Example: A company's profit is

$$P(x) = -0.1x^2 + 5x - 10$$

(in thousands \in) where x is production in thousands of units.

- Currently producing x = 20 thousand units
- Estimate profit if production increases to 21 thousand.

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Question: How would you estimate the profit change?

Guided Practice - 25 Minutes

Practice Set A: Quotient Rule

Work in pairs for 15 minutes

Differentiate the following using the quotient rule:

a)
$$f(x) = \frac{x^2+3}{x-1}$$

b)
$$g(x) = \frac{2x-5}{3x+2}$$

c)
$$h(x) = \frac{x^3}{x^2+1}$$

Practice Set B: Tangent Lines

Continue working in pairs for 10 minutes

A company's revenue function is $R(x) = 50x - 0.5x^2$ where x is the number of units sold (in hundreds).

- a) Find R'(x) and interpret its meaning.
- b) Find the equation of the tangent line to R(x) at x=30.
- c) Use the tangent line to estimate R(31).
- d) Compare your estimate to the actual value R(31).

Coffee Break - 15 Minutes

Business Applications

Marginal Analysis with Differentiation

A manufacturer has:

- Cost: $C(x) = 1000 + 50x + 0.5x^2$
- Revenue: $R(x) = 200x 0.5x^2$

where x is production in hundreds of units.

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Marginal functions:

- MC(x) = C'(x) = 50 + x
- MR(x) = R'(x) = 200 x
- Profit: $P(x) = R(x) C(x) = -x^2 + 150x 1000$
- MP(x) = P'(x) = -2x + 150

Profit Maximization

Question: At what production level is profit maximized?

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Set MR = MC:

$$200 - x = 50 + x \to 150 = 2x$$

x = 75 (hundreds of units)

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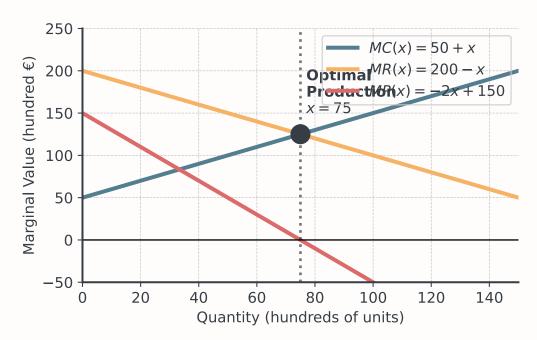
Optimal profit:

$$P(75) = -(75)^2 + 150(75) - 1000 = 4625 \ \text{hundred} \ \mathfrak{C}$$

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The maximum profit is €462,500 at 7,500 units.

Visualizing Marginal Analysis



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i Note

- For x < 75: MR > MC, so increasing production increases profit
- For x > 75: MR < MC, so increasing production decreases profit

Sensitivity Analysis Example

A software company's monthly user base U(t) (in thousands) follows:

$$U(t) = 10 + 2t + 0.1t^2$$

where t is months since launch.

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Revenue per user decreases with scale: r(u) = 20 - 0.05u

. . .

Total revenue: $R(t) = U(t) \cdot r(U(t))$

. . .

Question: How sensitive is revenue to time at t = 10 months?

Collaborative Problem-Solving

Challenge: Production Optimization

You're consulting for a manufacturing company. They produce specialty components with:

- Total Cost: $C(q) = 2000 + 40q + 0.2q^2$ (in €)
- Market Price: p = 100 0.1q (price decreases as quantity increases)
- Revenue: $R(q) = q \cdot p = q(100 0.1q) = 100q 0.1q^2$

where q is quantity in units.

Tasks

Work in groups of 3-4

- a) Find marginal cost MC(q) and marginal revenue MR(q) functions.
- b) Determine profit-maximizing production level.
- c) Calculate the actual maximum profit at this level.
- d) Find the tangent line to the profit function at the optimal point.
- e) The company is considering increasing production by 10 from the optimum. Use linear approximation to estimate the change in profit.
- f) Discuss: Why is MR = MC the condition for profit maximization? What happens if MR > MC or MR < MC?

Think-Pair-Share

Discussion Question

Think individually, then we discuss

Question: Consider two functions:

- Function A: f(x) = (x+1)(x+2)
- Function B: $g(x) = x^2 + 3x + 2$

When finding derivatives:

- 1. Are these functions the same?
- 2. Would you use different differentiation strategies?
- 3. Which approach is more efficient and why?

Wrap-Up & Key Takeaways

Today's Essential Rules

We covered quite some rules today:

Rule	Formula	When to Use
Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$	Any power of x
Constant Rule	$\frac{d}{dx}[c] = 0$	Constants
Sum Rule	(f+g)'=f'+g'	Sums/differences
Constant Multiple	(cf)' = cf'	Constants times functions
Product Rule	(uv)' = u'v + uv'	Products of functions
Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$	Quotients of functions

Rule Selection Strategy

Before differentiating, ask:

- 1. Can I simplify first? → Expand products, simplify fractions
- 2. Is it a sum/difference? → Differentiate term by term
- 3. Is it a simple power? \rightarrow Use power rule
- 4. Is it a product that can't simplify? \rightarrow Product rule
- 5. Is it a quotient? → Quotient rule (or rewrite as negative power)

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! Important

- Marginal analysis: Derivatives give marginal cost, revenue, profit
- Optimization: Set MR = MC to maximize profit
- ullet Sensitivity analysis: f'(a) shows how sensitive output is to input changes

Final Assessment

Quick Check

Work individually then we discuss

- 1. Find $\frac{d}{dx}[x^4 2x^2 + 7]$.
- 2. Differentiate $f(x) = x^2(3x+1)$ using the product rule.
- 3. What is the equation of the tangent line to $f(x) = x^3$ at x = 1?
- 4. True or False: The derivative of a quotient equals the quotient of derivatives.

Next Session Preview

Session 05-04

Chain Rule & Implicit Differentiation

• The Chain Rule: How to differentiate composite functions

- Combining techniques: Chain rule with product and quotient rules Implicit differentiation: Finding $\frac{dy}{dx}$ when you can't solve for y Related rates: How rates of change in different quantities are connected
- Business: Nested economic functions, inflation adjustments



Complete Tasks 05-03!