Session 05-02 - The Derivative as Rate of Change

Section 05: Differential Calculus

Dr. Nikolai Heinrichs & Dr. Tobias Vlćek

Entry Quiz - 10 Minutes

Quick Review from Session 05-01

Test your understanding of limits and continuity

1. Evaluate: $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

2. For $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 2x & \text{if } x > 2 \end{cases}$, is f continuous at $x = 2$?

3. A taxi charges €5 for trips up to 5km, then €8 for trips over 5km. Is the cost function continuous at 5km?

4. If $\lim_{x \to a} f(x) = L$, must $f(a) = L$? Explain.

Homework Discussion - 15 Minutes

Your questions from Session 05-01

What questions do you have regarding the tasks?

Learning Objectives

What You'll Master Today

- Understand rate of change as slope between points
- Develop the derivative from average to instantaneous rates
- Master derivative notation and terminology
- Apply marginal analysis to business decisions
- Interpret derivatives in economic contexts

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> ℹ **Today's Big Idea**
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> The derivative measures how things change right now, the instantaneous rate of change. This is the foundation of all calculus!

Part A: Average Rate of Change

Slope Review

Remember from linear functions: slope measures rate of change.

- Formula: $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$
- Interpretation: How much y changes per unit change in x
- Business meaning: Rate of growth, cost per unit, etc.

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Question:If revenue increases from €1000 to €1500 when production goes from 10 to 15 units, by how much does revenue increase per additional unit?

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Revenue increases by €100 per additional unit.
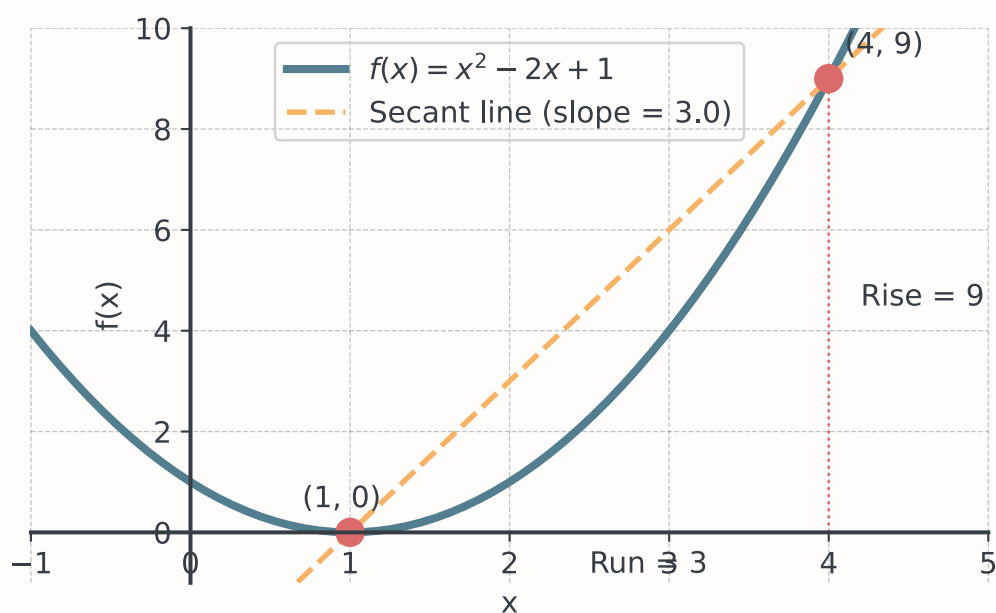
Average Rate of Change

For ANY function (not just linear), we can measure average change:

Average Rate of Change from a to b:

$$\text{ARC} = \frac{f(b) - f(a)}{b - a}$$

- This is the slope of the secant line through $(a, f(a))$ and $(b, f(b))$
- Tells us the average change over an interval
- For linear functions, ARC is constant
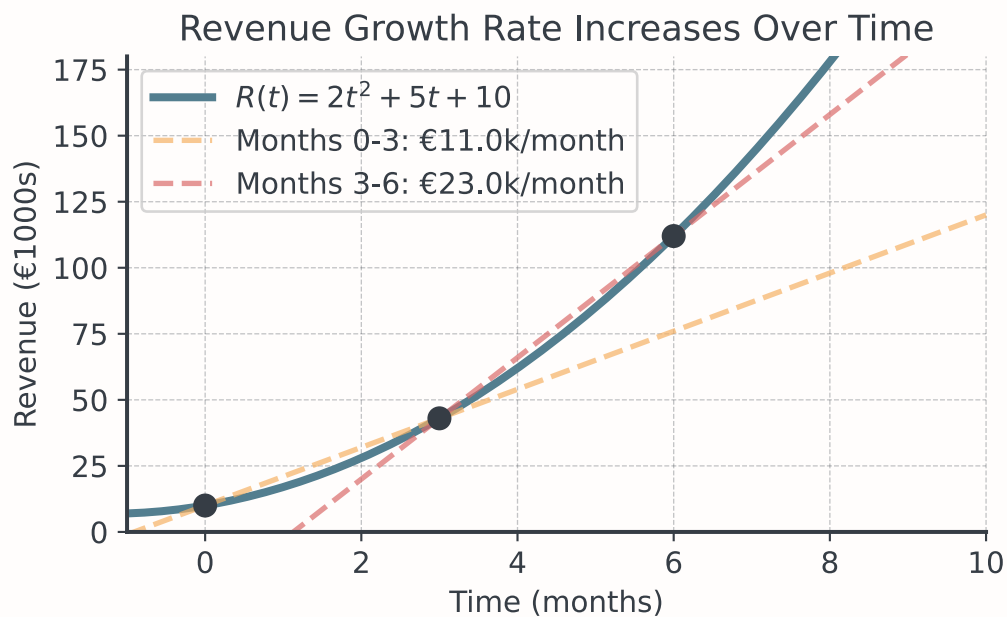- For nonlinear functions, ARC varies by interval

Visual: Secant Line

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Business Example: Revenue Growth

A company's monthly revenue (in €1000s): $R(t) = 2t^2 + 5t + 10$.

Revenue Growth Rate Increases Over Time

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Quick Practice - 10 Minutes

Calculate Average Rates

Work individually for 5 minutes

For $f(x) = x^2 + 3x$:

a) Find the average rate of change from $x = 1$ to $x = 3$

b) Find the average rate of change from $x = 1$ to $x = 2$

c) Find the average rate of change from $x = 1$ to $x = 1.5$

d) What pattern do you notice as the interval gets smaller?

Break - 10 Minutes

Part B: Instantaneous Rate of Change

The Limit Process

What if we want the rate of change at a single point?

- Average rate needs TWO points
- At ONE point, there's no interval to measure over
- Solution: Make the interval infinitesimally small!
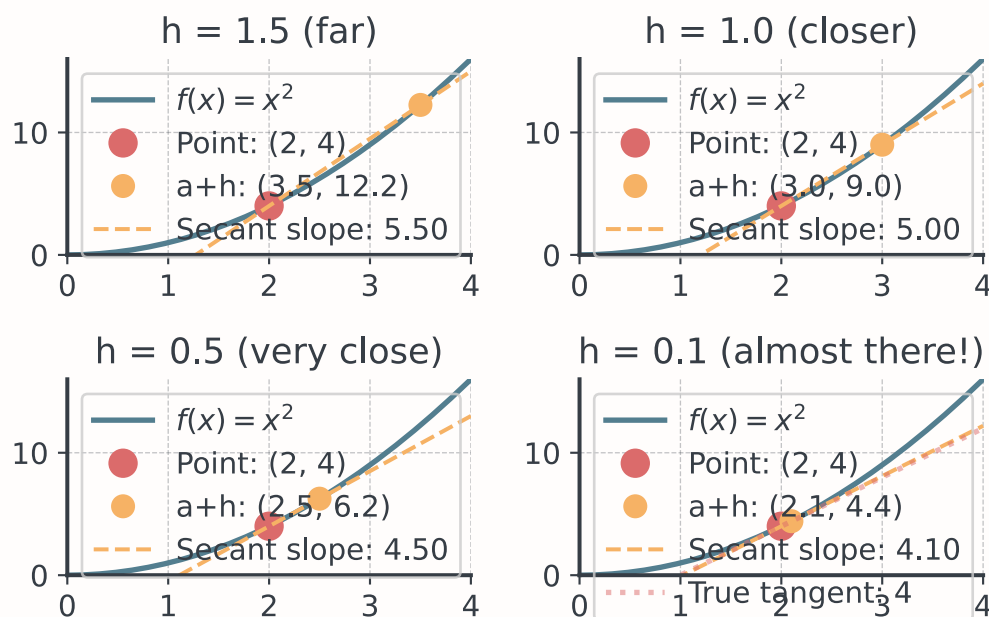- Use limits to let the second point approach the first

. . .

Instantaneous Rate of Change at $x = a$: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

. . .

This limit (if it exists) is called the derivative of f at a.

Visual: Secant Becoming Tangent

h = 1.5 (far)

$f(x) = x^2$
Point: (2, 4)
a+h: (3.5, 12.2)
Secant slope: 5.50

h = 1.0 (closer)

$f(x) = x^2$
Point: (2, 4)
a+h: (3.0, 9.0)
Secant slope: 5.00

h = 0.5 (very close)

$f(x) = x^2$
Point: (2, 4)
a+h: (2.5, 6.2)
Secant slope: 4.50

h = 0.1 (almost there!)

$f(x) = x^2$
Point: (2, 4)
a+h: (2.1, 4.4)
Secant slope: 4.10
True tangent

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> **!** Important
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> As $h \to 0$, the secant line approaches the tangent line!

The Derivative Definition

The derivative of f at $x = a$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Geometric meaning: Slope of the tangent line at $(a, f(a))$
- Physical meaning: Instantaneous rate of change
- Verbal: "f prime of a" or "the derivative of f at a"

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> ⚠️ **Warning**
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> The derivative is the LIMIT of average rates of change, not just "a very small" average rate!

Computing Derivative from Definition

Example: Find $f'(2)$ if $f(x) = x^2$.

1. Definition: $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$
2. Substitute: $= \lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$
3. Expand & factor: $= \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} (4 + h)$
4. Evaluate: $= 4 \to$ The tangent slope at $x = 2$ is 4

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> ℹ️ **Note**
>
> Not too complicated, right?

Part C: Derivative Notation & Terminology

The Derivative as a Function

When $f'(x)$ (not $f'(a)$), we mean the derivative at ANY point x:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- This defines a NEW function f' from the original function f
- $f'(x)$ gives the slope of the tangent at each x

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Example: If $f(x) = x^2$, then $f'(x) = 2x$ (we'll prove this soon!)

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- At $x = 1$: slope is $2(1) = 2$ and at $x = 3$: slope is $2(3) = 6$
- The function gets steeper as x increases

Multiple Notations

There are several ways to write "the derivative of f at x":

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Notation	Read as	Context
$f'(x)$	"f prime of x"	Function notation
$\frac{df}{dx}$	"derivative of f with respect to x"	Leibniz notation
$\frac{dy}{dx}$	"dy dx"	When y = f(x)
$\frac{d}{dx}[f(x)]$	"d dx of f of x"	Operator notation

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> 💡 Tip
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> We'll primarily use $f'(x)$ for general work and $\frac{dy}{dx}$ when emphasizing the variables.

Differentiability vs Continuity

Not every function has a derivative everywhere!

- If $f'(a)$ exists, we say f is differentiable at a
- Theorem: If f is differentiable at a, then f is continuous at a
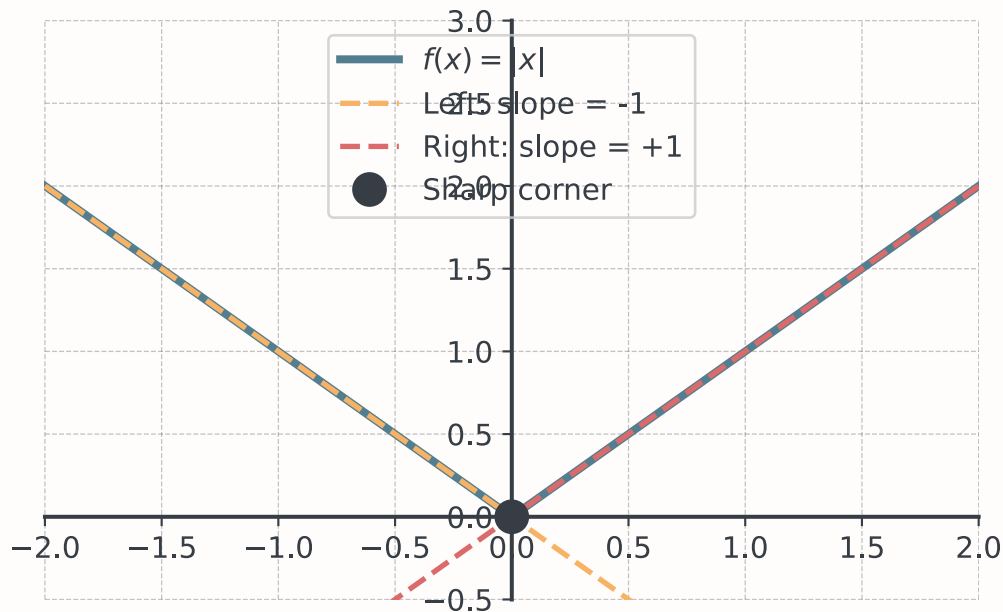- Converse is false: Continuous functions may not be differentiable

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Question: Ever hear the word Theorem before?

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A Theorem is basically a statement that has been proven to be true.

Continuous but not differentiable

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> **!** **Important**
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> At $x = 0$, the limit doesn't exist (left and right slopes differ).

Practice: Derivatives from Definition

Work in pairs for 15 minutes

Use the limit definition to find $f'(a)$ at the given point:

a) $f(x) = 3x + 2$ at $a = 1$

b) $f(x) = x^2 - 4x$ at $a = 3$

c) $f(x) = \frac{1}{x}$ at $a = 2$

d) Interpret each derivative: what does the value tell you?

Coffee Break - 15 Minutes

Part D: Marginal Analysis in Economics

The Marginal Concept

In economics, "marginal" means "the next one":

- Marginal cost: Cost of producing one more unit
- Marginal revenue: Revenue from selling one more unit

- Marginal profit: Profit from one more unit

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Question: How can we apply this now with what we've learned?

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> **! Important**
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> The derivative tells us the instantaneous rate of increase in cost per unit of production.

Why Derivatives Give Marginal Values

True marginal cost at $x = 100$ units:

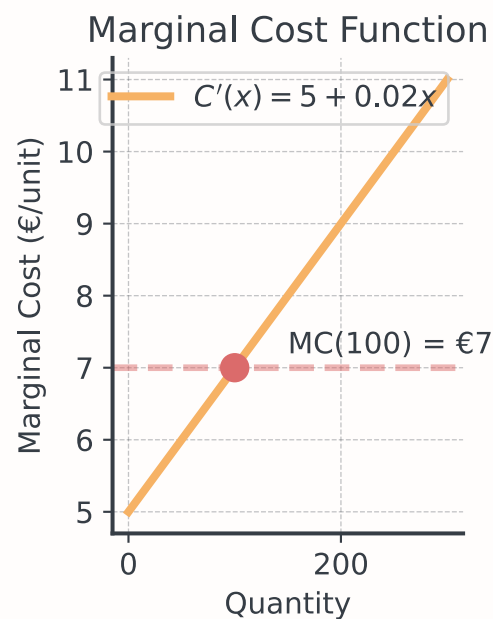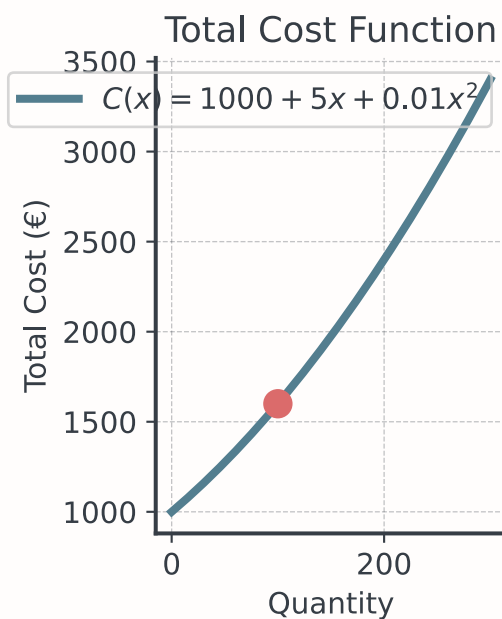$$MC(100) = C(101) - C(100)$$

But this is approximately:

$$C'(100) = \lim_{h \to 0} \frac{C(100 + h) - C(100)}{h}$$

When $h = 1$ (one more unit), the approximation is very good:

$$C'(100) \approx \frac{C(101) - C(100)}{1} = C(101) - C(100)$$

Example: Marginal Cost

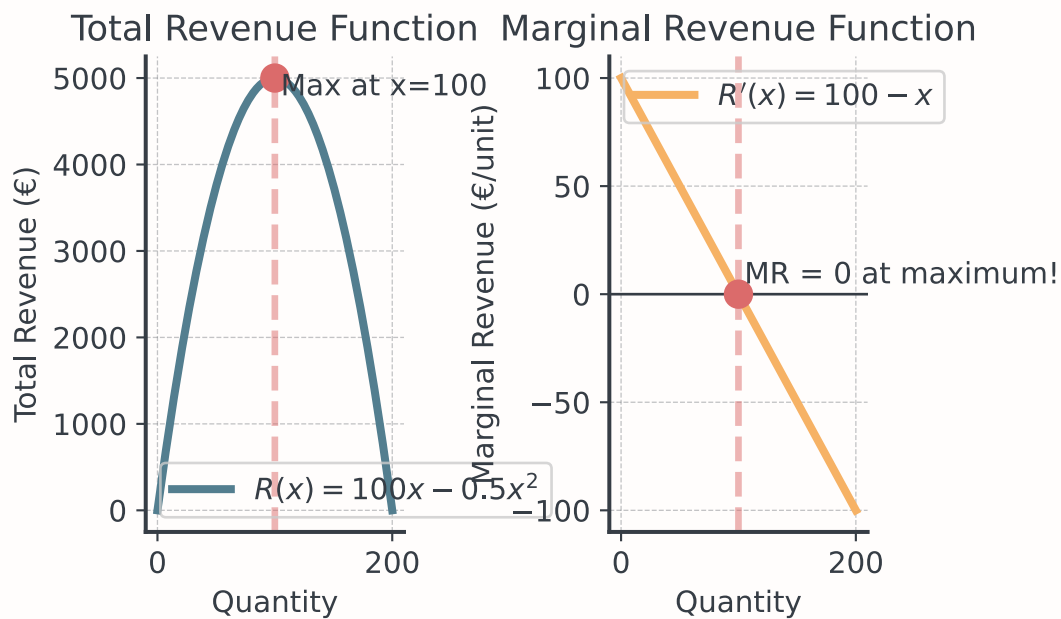A factory's total cost function is: $C(x) = 1000 + 5x + 0.01x^2$.

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Marginal Revenue

Revenue: $R(x) = x \cdot p = x(100 - 0.5x) = 100x - 0.5x^2$

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Marginal Profit

Since $P(x) = R(x) - C(x)$, we have:

$$P'(x) = R'(x) - C'(x)$$

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Interpretation: Marginal profit = Marginal revenue – Marginal cost

- Produce more if $MR > MC$ (next unit adds more revenue than cost)
- Produce less if $MR < MC$ (next unit costs more than it earns)
- Optimal: $MR = MC$ (marginal revenue equals marginal cost)

. . .

> 💡 **Tip**
>
> Makes sense, right? Profit is maximized when $MR = MC$!

Practice Set: Marginal Analysis I

Work for 5 minutes

Use the given derivatives to answer each question:

1. Cost function $C(x) = 500 + 3x + 0.02x^2$
   - Given: $C'(x) = 3 + 0.04x$
   - Calculate $C'(50)$ and interpret what it means for the business
2. Revenue function $R(x) = 80x - 0.5x^2$
   - Given: $R'(x) = 80 - x$
   - At what quantity does $R'(x) = 0$? What does this mean?

Practice Set: Marginal Analysis II

Work for 5 minutes

Use the given derivatives to answer each question:

3. Given: $P'(x) = R'(x) - C'(x) = 77 - 1.04x$
   - What production level maximizes profit (set $P'(x) = 0$)?
4. Interpretation: If $C'(100) = 7$ and $R'(100) = 5$, should the company increase or decrease production?

Collaborative Problem-Solving

Coffee Shop Economics

A coffee shop's daily profit depends on the number of drinks sold:

$$P(x) = -0.02x^2 + 8x - 200$$

where x is the number of drinks and P is profit in euros.

a) Find the average rate of change of profit from 100 to 150 drinks.

b) Estimate $P'(150)$ by computing $\frac{P(151) - P(150)}{1}$.

c) Interpret: What does $P'(150)$ tell the shop owner?

d) Determine: Should they try to sell more than 150 drinks?

e) Find where $P'(x) = 0$ by solving $8 - 0.04x = 0$.

f) Explain the business significance of the point where $P'(x) = 0$.

Think-Pair-Share

Conceptual Question

Think → Discuss → Share

Question: A company finds that $C'(500) = 12$ and $R'(500) = 15$.

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What should they do?

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a) Increase production
b) Decrease production
c) Keep production at 500 units

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Explain your reasoning using marginal analysis.

Wrap-Up & Key Takeaways

Today's Essential Concepts

- Average rate of change is the slope of a secant line
- Instantaneous rate is the limit as the interval shrinks to zero
- The derivative $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$
- Marginal analysis: Derivatives give marginal cost, revenue, profit
- Optimization: Maximize profit where $MR = MC$ (where $P'(x) = 0$)

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> ! Important
>
> The derivative is THE fundamental tool in calculus. It measures instantaneous rate of change. It tells us how things are changing right now!

Final Assessment

Quick Check

Work individually, then we discuss

1. If $f(3) = 7$ and $f(3.1) = 7.5$, estimate $f'(3)$.

2. A cost function has $C'(200) = 8$. What does this tell you?

3. For profit maximization, we set _____ = _____.

4. True or False: If a function is continuous at $x = a$, it must be differentiable at $x = a$.

Next Session Preview

Coming Up

Differentiation Rules

- Power rule, sum rule, constant multiple rule
- Product and quotient rules
- Finding equations of tangent lines
- More business applications
- NO MORE tedious limit calculations!

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> 💡 Tip
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> Complete Tasks 05-02!