Rational & Logarithmic Functions

Session 04-05: Advanced Function Analysis

Mathematics for Business Students

Entry Quiz - 10 Minutes

Review from Session 04-04

Work individually for 5 minutes, then we discuss

1. Transformations: If $g(x) = 2f(x-3)+1$, describe all transformations from $f(x)$.
2. Logarithms: Simplify $\log_2(8x)$ using logarithm properties.
3. Rational Behavior: What happens to $\frac{1}{x}$ as $x \to 0^+$? As $x \to 0^-$?
4. Exponential Equation: Solve $2^{x-1} = 16$.

Homework Discussion - 15 Minutes

Your Questions from Tasks 04-04

Let's discuss the problems you found challenging

Learning Objectives

Today's Goals

By the end of this session, you will be able to:

- Analyze rational functions completely (asymptotes, holes, intercepts)
- Understand logarithmic properties and transformations
- Master semi-log and log-log scales
- Model business scenarios with average cost functions
- Interpret exponential growth using logarithmic scales
- Solve complex equations involving logs and rationals

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> **ℹ Note**
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> This session connects algebra with real business applications!

Rational Functions Deep Dive

Structure of Rational Functions

A rational function has the form:

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$

1. Domain: All real numbers except where $Q(x) = 0$
2. Zeros: Where $P(x) = 0$ (and $Q(x) \neq 0$)
3. Vertical Asymptotes: Where $Q(x) = 0$ (canceling common factors)
4. Holes: Where both $P(x) = 0$ and $Q(x) = 0$ (canceled factors)
5. Horizontal/Oblique Asymptotes: Determined by degree comparison

What Are Asymptotes?

An asymptote is a line a function approaches

- Think of it like a boundary the graph gets infinitely close to
- Vertical asymptotes: Never crossed or touched (undefined there)
- Horizontal/oblique asymptotes: Can be crossed at finite x-values, but approached as $x \to \pm\infty$
- Three types: vertical, horizontal, and oblique (slanted)

Vertical Asymptotes

Occur where the denominator equals zero (and numerator doesn't)

Mathematical definition:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty$$

- The function "blows up" (goes to ∞ or $-\infty$)
- Graph has a vertical line at $x = a$
- Function is undefined at this point
- Example: $f(x) = \frac{1}{x}$ has vertical asymptote at $x = 0$

Horizontal Asymptotes

Describe the end behavior as $x \to \pm\infty$

Three cases based on degrees of $\boldsymbol{P(x)}$ and $\boldsymbol{Q(x)}$:

Degree: (P) < (Q)

$$f(x) = \frac{2x + 1}{x^3 - 5}$$

- Denominator grows faster
- Horizontal asymptote: $y = 0$

- The function approaches zero

Degree: (P) = (Q)

$$f(x) = \frac{3x^2 + 2x - 1}{2x^2 + 5}$$

- Both grow at same rate
- Horizontal asymptote: $y = \frac{3}{2}$ (ratio of leading coefficients)

Degree: (P) > (Q)

$$f(x) = \frac{x^3 + 2x}{x^2 - 1}$$

- Numerator grows faster
- No horizontal asymptote
- May have an oblique (slanted) asymptote instead

Oblique (Slanted) Asymptotes

When degree of P exceeds degree of Q by exactly 1

1. Perform polynomial long division[1]: $\frac{P(x)}{Q(x)} = L(x) + \frac{R(x)}{Q(x)}$
2. The quotient $L(x)$ (a linear function) is the oblique asymptote
3. As $x \to \pm\infty$, the remainder term $\frac{R(x)}{Q(x)} \to 0$

...

Example: $f(x) = \frac{x^2+1}{x-1} = \frac{x^2+1}{x-1} = x + 1 + \frac{2}{x-1}$

...

Oblique asymptote: $\boldsymbol{y = x + 1}$

Holes vs. Asymptotes

Critical distinction when factors cancel!

Hole

- Factor appears in both numerator and denominator
- Example: $f(x) = \frac{(x-2)(x+1)}{(x-2)(x+3)}$
- Factor $(x - 2)$ cancels
- Hole at $\boldsymbol{x = 2}$, not an asymptote!
- Simplified: $f(x) = \frac{x+1}{x+3}, x \neq 2$

Vertical Asymptote

- Factor appears only in denominator
- $f(x) = \frac{x+1}{x+3}$

[1] No worries, no need to learn long division. This is just for the sake of completeness.

- Factor $(x + 3)$ doesn't cancel
- Vertical asymptote at $x = -3$
- Function undefined, goes to $\pm\infty$

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> **! Important**
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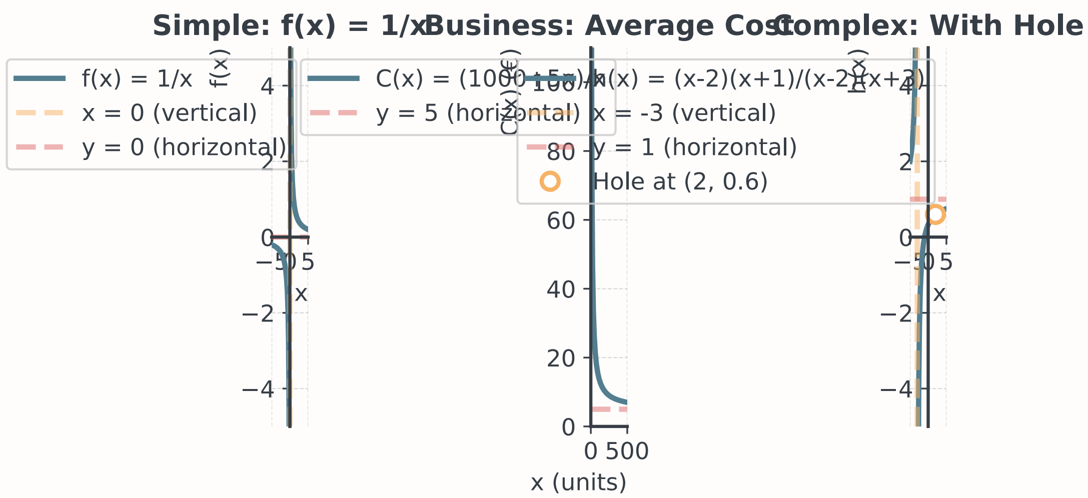
> Always factor completely and cancel common factors before identifying asymptotes!

Examples for Rational Functions



Asymptote Rules

Systematic Approach for Finding Asymptotes

Step 1: Factor completely

$$f(x) = \frac{P(x)}{Q(x)} = \frac{\text{factored form}}{\text{factored form}}$$

Step 2: Cancel common factors $\longrightarrow$ These create holes

Step 3: Vertical asymptotes $\longrightarrow$ Remaining factors in denominator

Step 4: Horizontal/Oblique asymptotes $\longrightarrow$ Compare degrees

Asymptote Analysis Challenge

3 minutes individual, 2 minutes pair discussion, 2 minutes class share

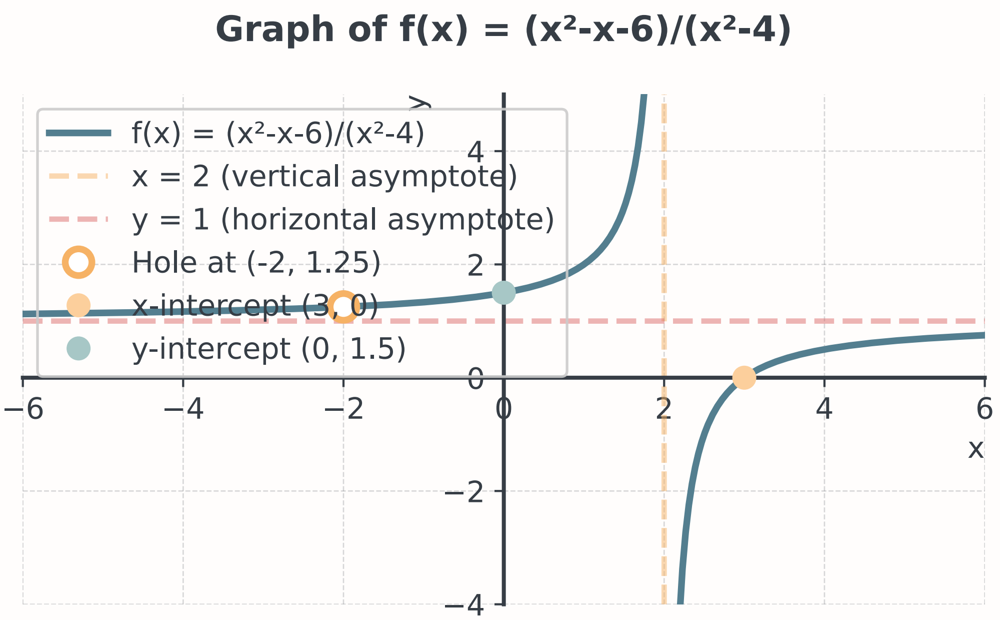
Analyze the function: $f(x) = \frac{x^2 - x - 6}{x^2 - 4}$

Your tasks:

1. Factor numerator and denominator

2. Identify any holes
3. Find all asymptotes
4. Determine x and y intercepts
5. Sketch a rough graph

Asymptote Analysis



Break - 10 Minutes

Business Application - Average Cost

Average Cost Functions

In business, the average cost per unit is:

$$AC(x) = \frac{\text{Total Cost}}{\text{Quantity}} = \frac{C(x)}{x} = \frac{F + vx}{x} = \frac{F}{x} + v$$

- F = Fixed costs
- v = Variable cost per unit
- x = Number of units

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> ! Important
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> Do you get the idea here?

Key Properties

These functions often have the same properties:

- Vertical asymptote at $x = 0$
- Horizontal asymptote at $y = v$
- Always decreasing for $x > 0$ (economies of scale)
- Minimum average cost approaches v as $x \to \infty$

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> 💡 Tip
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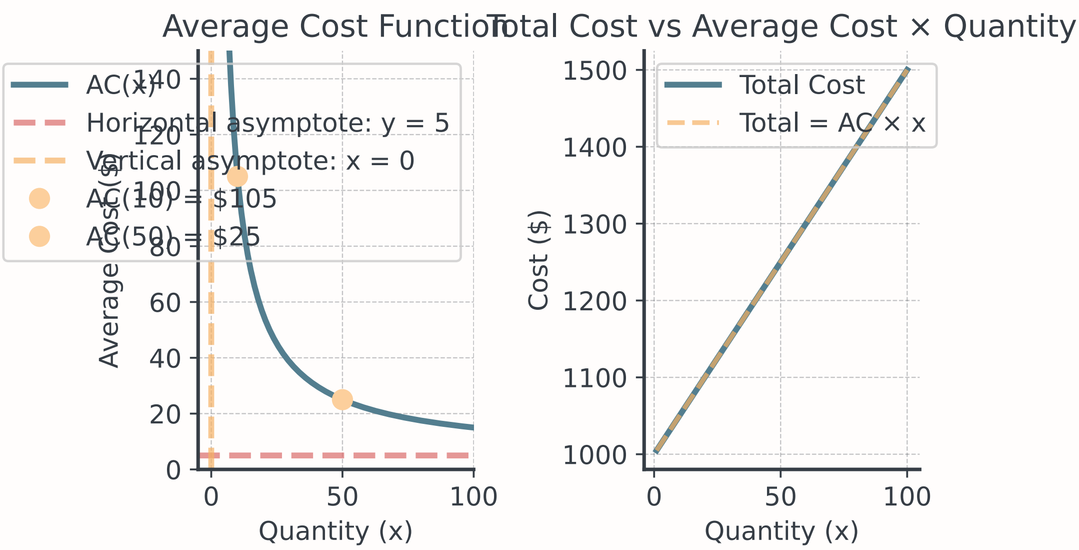
> Let's see an example!

Visualization of Average Cost



Manufacturing Analysis

Work through this business scenario and then we compare

A company has fixed costs of $5000 per month and variable costs of $20 per unit.

a) Write the average cost function
b) Find the horizontal asymptote and interpret it
c) How many units minimize average cost to within $5 of the minimum?
d) Graph the function

Logarithmic Functions

Recap: Logarithm Properties

The Big Three Rules for Logarithms

1. Product Rule: $\log_b(xy) = \log_b(x) + \log_b(y)$
2. Quotient Rule: $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
3. Power Rule: $\log_b(x^n) = n \cdot \log_b(x)$

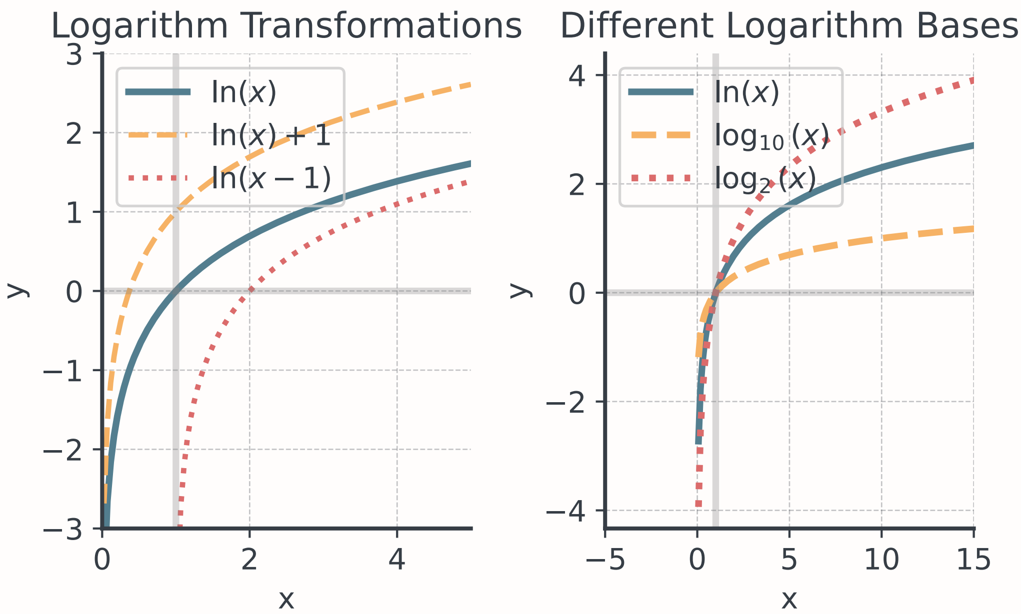
...

Special Values

- $\log_b(1) = 0$ for any base b
- $\log_b(b) = 1$
- $\log_b(b^n) = n$
- $b^{\log_b(x)} = x$

Logarithmic Transformations

```
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_36143/2386517576.py:14: RuntimeWarning: invalid value encountered
in log
  ax1.plot(x, np.log(x), color=BRAND_COLORS["twoDark"], linewidth=2.5,
label='$\\ln(x)$')
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_36143/2386517576.py:15: RuntimeWarning: invalid value encountered
in log
  ax1.plot(x, np.log(x) + 1, '--', color=BRAND_COLORS["oneDark"],
linewidth=2, label='$\\ln(x) + 1$')
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_36143/2386517576.py:16: RuntimeWarning: invalid value encountered
in log
  ax1.plot(x, np.log(x - 1), ':', color=BRAND_COLORS["threeDark"],
linewidth=2, label='$\\ln(x-1)$')
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_36143/2386517576.py:30: RuntimeWarning: invalid value encountered
in log
  ax2.plot(x2, np.log(x2), color=BRAND_COLORS["twoDark"], linewidth=2.5,
label='$\\ln(x)$')
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_36143/2386517576.py:31: RuntimeWarning: invalid value encountered
in log10
  ax2.plot(x2, np.log10(x2), '--', color=BRAND_COLORS["oneDark"],
linewidth=2.5, label='$\\log_{10}(x)$')
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_36143/2386517576.py:32: RuntimeWarning: invalid value encountered
in log2
  ax2.plot(x2, np.log2(x2), ':', color=BRAND_COLORS["threeDark"],
linewidth=2.5, label='$\\log_2(x)$')
```



Spot the Error: Logarithm Mistakes

Find and fix the errors!

Problem: Solve $\log_2(x) + \log_2(x-2) = 3$

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Student Solution:

"$\log_2(x) + \log_2(x-2) = 3$

$\log_2(x + x - 2) = 3$

$\log_2(2x-2) = 3$

$2x - 2 = 8$

$x = 5$"

Semi-log and Log-log Plots

Semi-log Plot (y-axis log)

When data spans several orders of magnitude:

- Exponential growth/decay patterns
- Compound interest, population growth

. . .

Recognition:

- Exponential functions appear as straight lines
- Slope represents growth rate

Log-log Plot (both axes log)

When working with power law relationships:

- Power law relationships
- Allometric scaling
- Economic relationships (supply/demand curves)

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Recognition:

- Power functions $y = ax^b$ appear as straight lines
- Slope equals the exponent b

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> **i Note**
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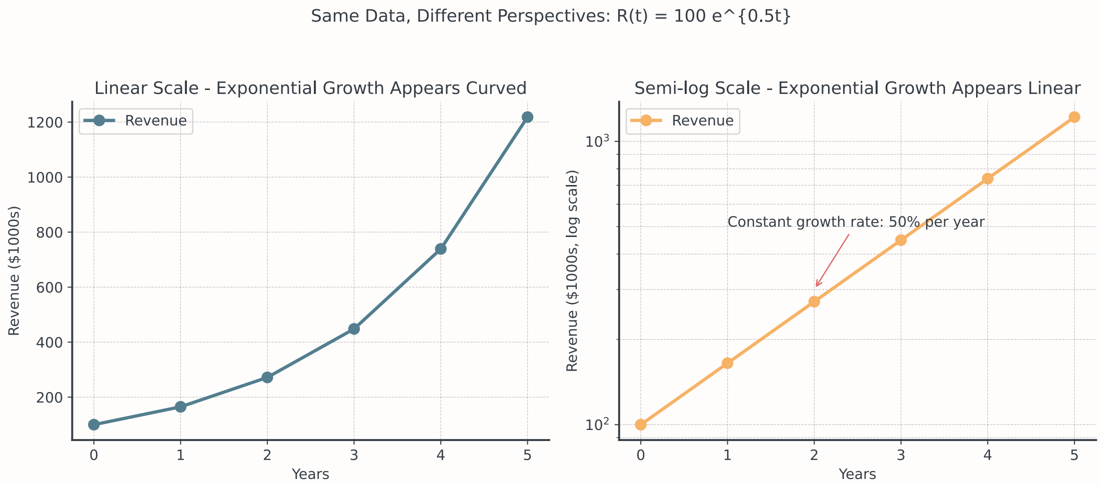
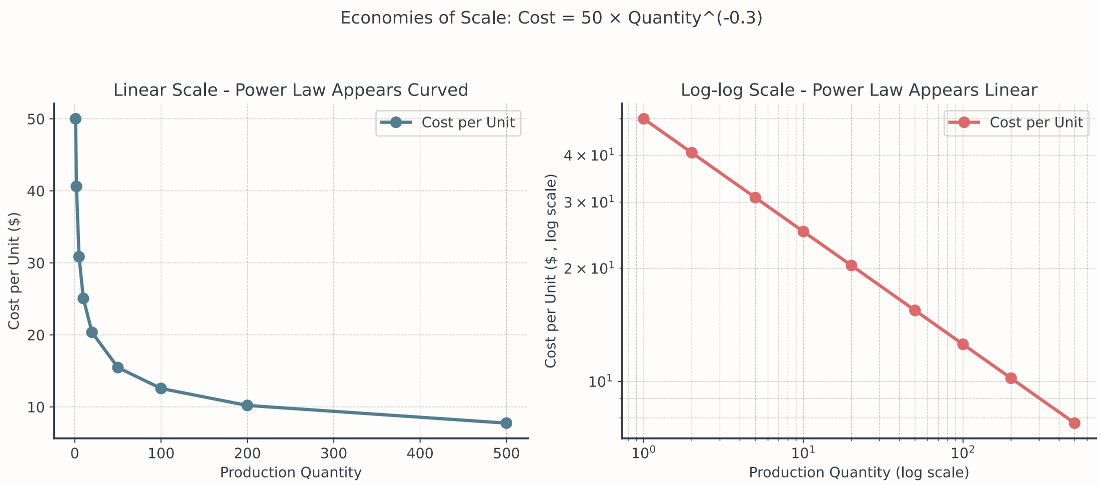
> Let's viusalize both!

Business Application: Market Analysis



Power Law Example: Production Costs



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> **Tip**
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> On the log-log plot, the straight line confirms a power law relationship. The slope of −0.3 means that doubling production reduces per-unit cost by about 19% (2^(-0.3) ≈ 0.81).

Guided Practice - 20 Minutes

Complex Problems

Work individually, then check with class

1. Rational Function: Analyze completely $f(x) = \frac{2x^2-8}{x^2-x-2}$
2. Logarithmic Equation: Solve $\ln(x+3) - \ln(x-1) = \ln(2)$
3. Multi-Step Challenge: A factory's efficiency rating (as %) depends on production volume x (in thousands of units per month, $x > 0$).

$$E(x) = \frac{100(x^2 - 4x)}{x^2 - 8x + 20}$$

- Factor the numerator and find all zeros
- Find all asymptotes and interpret their meaning

Coffee Break - 15 Minutes

Synthesis & Applications

Real-World Application: pH Scale

Where logarithmic properties are important:

$$\mathrm{pH} = -\log_{10}[\mathrm{H}^+]$$

where $[\mathrm{H}^+]$ is hydrogen ion concentration in mol/L

- Domain: $(0, \infty)$ for concentration
- Range: Typically 0-14 for pH
- Each unit change = 10× concentration change
- pH 7 is neutral ($[\mathrm{H}^+] = 10^{-7}$)

Orange Juice

If orange juice has pH = 3.5:

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Question: How much more acidic is orange juice compared to neutral water?

. . .

$$3.5 = -\log_{10}[\mathrm{H}^+]$$

$$[\mathrm{H}^+] = 10^{-3.5} \approx 3.16 \times 10^{-4} \quad \mathrm{mol/L}$$

. . .

This is 1000 times more acidic than neutral water!

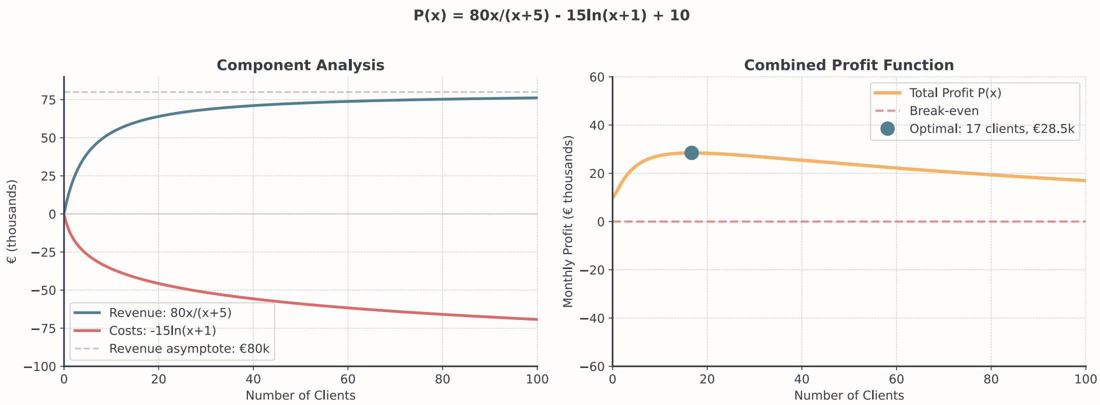
Profit with Components I

A firm's profit depends on portfolio size x (number of clients):

$$P(x) = \frac{80x}{x+5} - 15\ln(x+1) + 10$$

- Rational term $\frac{80x}{x+5}$: Revenue per client (approaches €80k asymptote)
- Logarithmic term $-15\ln(x+1)$: Overhead & complexity costs
- Fixed term $+10$: Base profit offset
- Key insight: Logarithmic costs eventually erode the revenue gains
- Critical question: What's the optimal client portfolio size?

Profit with Components II



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> **! Important**
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> Initially, revenue growth outpaces cost growth ⟶ profits increase. Eventually, costs catch up and overtake revenue ⟶ profits decline!

Tasks - 15 Minutes

Function Investigation

Analyzes function types and then we dicuss

a) $f(x) = \frac{x^2-9}{x-3}$
b) $AC(x) = \frac{2000+15x+0.01x^2}{x}$
c) $g(t) = 50\ln(t+2) - 100$
d) $h(x) = \frac{100}{x+5} + \ln(x)$

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1. Find domain and range
2. Identify all asymptotes/discontinuities
3. Describe a business application

Session Wrap-Up

Key Takeaways

- Systematic asymptote finding through factoring
- Distinguishing holes from vertical asymptotes
- Business applications with average cost functions
- Semi-log and log-log plots for data analysis
- Rational functions model constrained optimization
- Logarithms linearize exponential and power relationships

- Combined functions capture complex business scenarios

Final Assessment - 10 Minutes

Quick Check

Work individually to test your understanding

1. Rational Function: Find all asymptotes and holes for: $f(x) = \frac{x^2-1}{x^2+x-2}$
2. Logarithmic Equation: Solve: $2\ln(x) - \ln(x+6) = \ln(4)$

Homework Preview

Tasks 04-06
You'll practice:

1. Complete rational function analysis (find all features)
2. Logarithmic equation solving using properties
3. Business optimization with average cost
4. Data interpretation with different scales
5. Synthesis problems combining both function types

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> Tip
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> Success Strategy: Always factor rational functions first as it reveals everything!

Mock Exam Strategies

What's important for the Mock Exam

- Read carefully: Every word and number matters
- Show all work: Partial credit is available for clear methodology
- Label everything: Variables, units, and graph features
- Check domains: Especially for logarithmic and rational functions
- Verify solutions: Substitute back when possible

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> Note
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> This exam tests your mastery functions. Apply the systematic methods we've practiced!