Session 04-04 - Introduction to Trigonometric Functions

Section 04: Advanced Functions

Dr. Nikolai Heinrichs & Dr. Tobias Vlćek

Entry Quiz - 10 Minutes

Review from Session 04-03

Work individually for 5 minutes, then we discuss

1. Evaluate: $e^{\ln(5)}$
2. If an investment grows from €1000 to €2000 in 8 years with continuous compounding, what is the annual rate r? (Use $A = Pe^{rt}$)
3. Solve: $2^{3x-1} = 64$
4. A bacteria population doubles every 4 hours. If you start with 100 bacteria, write the exponential model $N(t)$ where t is in hours.

Homework Discussion - 15 Minutes

Your questions from Tasks 04-03

Focus on exponential functions and applications

- Challenges with exponential growth and decay models
- Compound interest calculations (discrete vs. continuous)
- Half-life and doubling time problems
- Comparing exponential vs. polynomial growth rates
- Real-world modeling (population, finance, radioactive decay)

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> **ℹ Note**
>
> Trigonometry introduces periodic (repeating) behavior - a new type of function compared to the always-increasing exponentials! Instead of continuous growth, we'll see cycles and oscillations.

Learning Objectives

Today's Goals

By the end of this session, you will be able to:

- Understand angles in degrees and radians
- Define sine, cosine, and tangent using the unit circle
- Calculate exact values for special angles
- Sketch basic trigonometric function graphs
- Identify amplitude, period, and phase shifts
- Apply trigonometry to real-world periodic phenomena

Angles and Their Measurement

Degrees vs. Radians

Two ways to measure angles

Degrees

- Full rotation = 360°
- Right angle = 90°
- Straight angle = 180°
- Historical: Based on ancient calendars

Radians

- Full rotation = 2π radians
- Right angle = $\frac{\pi}{2}$ radians
- Straight angle = π radians
- Radians make calculus formulas simpler!
- They're the "natural" unit for mathematics

. . .

! Important

Conversion: $180° = \pi \text{ radians}$, $1° = \frac{\pi}{180}$ radians and $1 \text{ radian} = \frac{180°}{\pi}$

Why Radians Are Natural

The arc length connection

For a circle with radius r and central angle θ (in radians):

$$\text{Arc length } \ s = r\theta$$

. . .

Why is this great?

- If $\theta = 1$ radian, then arc length = radius (that's the definition!)
- For a full circle: $s = r \cdot 2\pi = 2\pi r$ (the circumference formula!)
- No conversion factors needed - it just works!

The Unit Circle

Defining the Unit Circle I

The unit circle is a circle with:

- Center at the origin (0, 0)
- Radius = 1
- Equation: $x^2 + y^2 = 1$

. . .

For any angle θ from the positive x-axis:

- The point on the circle is $(\cos\theta, \sin\theta)$
- This is the fundamental definition!

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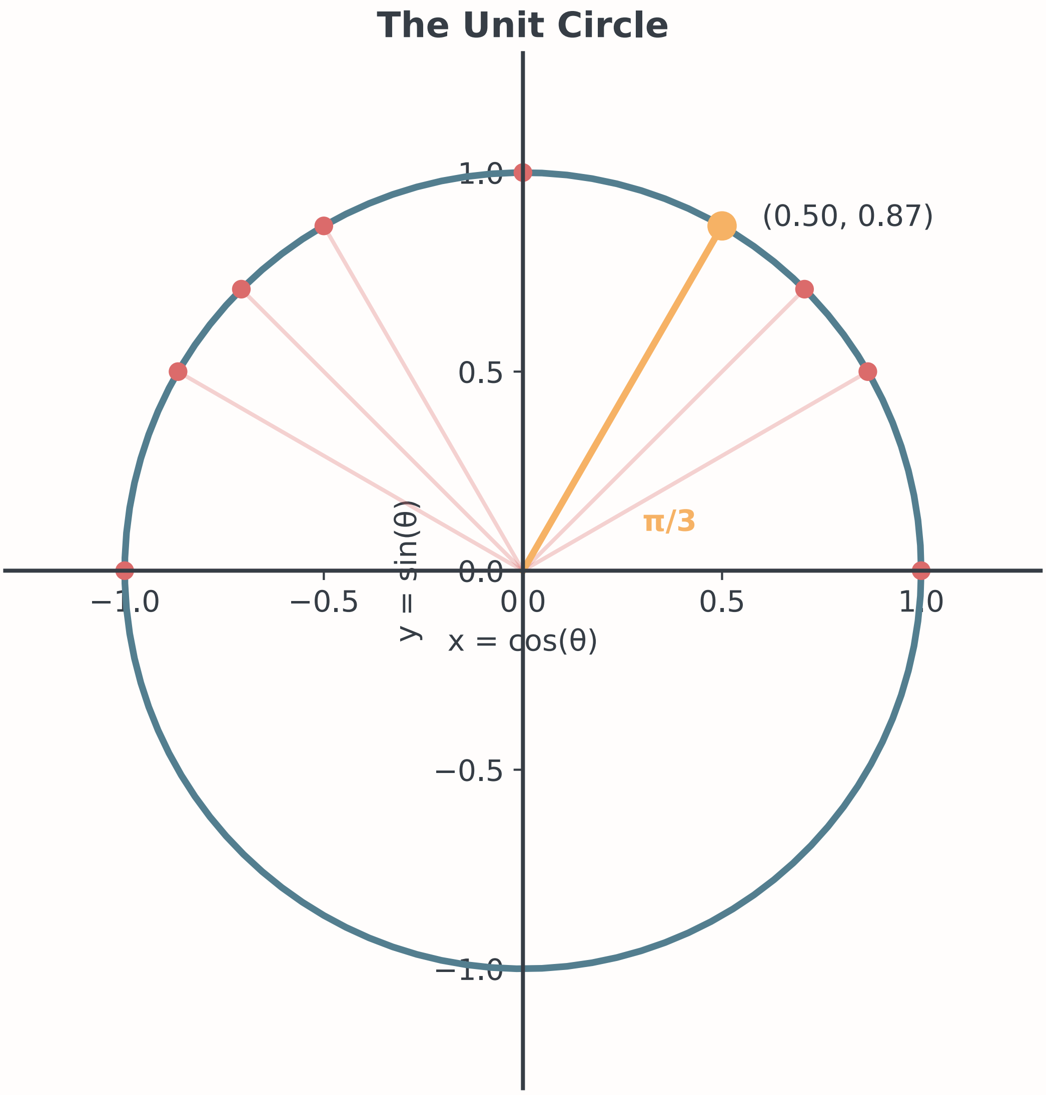
> ! Important
>
> Every point on the unit circle can be written as $(cos\theta, sin\theta)$ for some angle θ!

Defining the Unit Circle II



Think-Pair-Share: Unit Circle Practice

2 minutes individual, 3 minutes pairs, 2 minutes class discussion

Find the coordinates on the unit circle

For each angle, find the point $(\cos \theta, \sin \theta)$:

1. $\theta = \pi/2$
2. $\theta = \pi$
3. $\theta = 3\pi/2$
4. $\theta = 2\pi$

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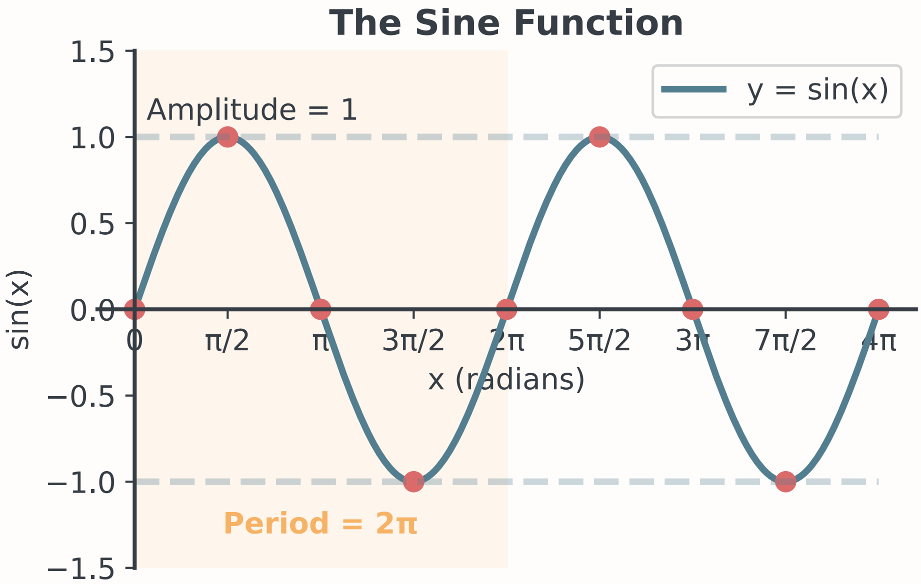
Discuss: What pattern do you notice as we go around the circle?

The Sine and Cosine Functions

Sine Function Graph



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> 💡 Tip
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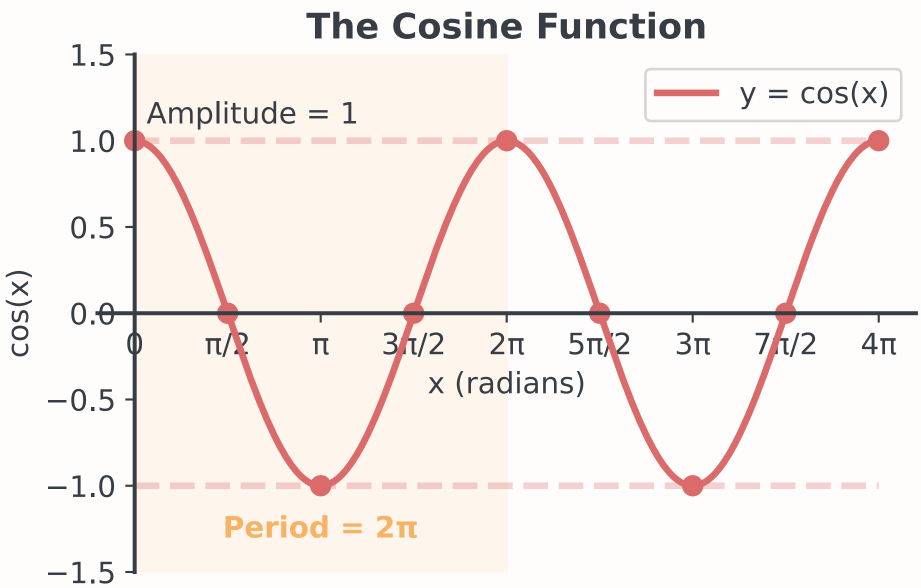
> Domain: All real numbers, range: [–1, 1], period: 2π

Cosine Function Graph



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> 💡 Tip
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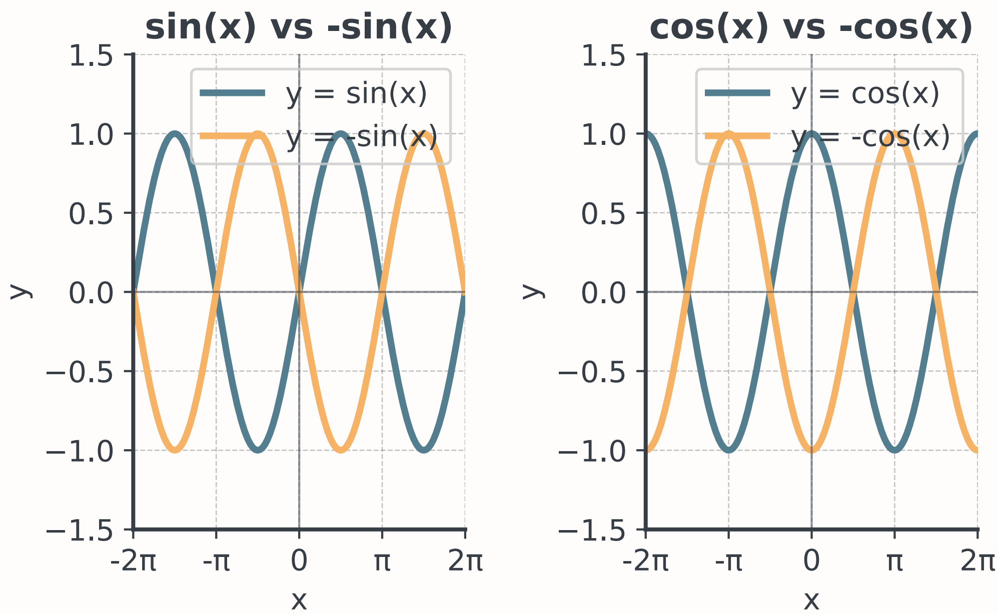
> Domain: All real numbers, range: [−1, 1], period: 2π -> Shifted by $\pi/2$

Negative Transformations

Understanding -sin(x) and -cos(x)



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> 💡 Tip
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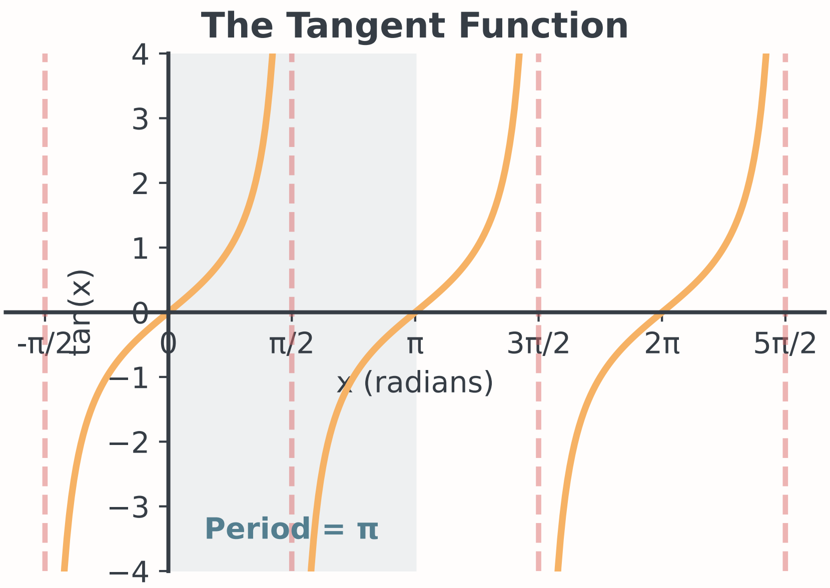
> Multiplying by −1 creates a reflection across the x-axis!

The Tangent Function

Definition and Graph

The ratio that creates asymptotes



Why Tangent Matters

Understanding slopes and angles

The tangent function has a special geometric meaning:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

. . .

- Navigation & Surveying: Finding heights of buildings or mountains
- Physics: Calculating angles of projectile motion or inclined planes
- Computer Graphics: Rotating objects and calculating viewing angles

. . .

> Tip
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> However, it is likely not important for the FSP and thus we won't go into too much detail here!

Amplitude and Period

Transformations of Sine and Cosine I

Modifying the basic wave

General form: $y = A\sin(B(x - C)) + D$

- A: Amplitude (height of wave)
- B: Affects period (Period = 2π/B)

- C: Phase shift (horizontal shift)
- D: Vertical shift

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> 💡 You already know the order from functions!
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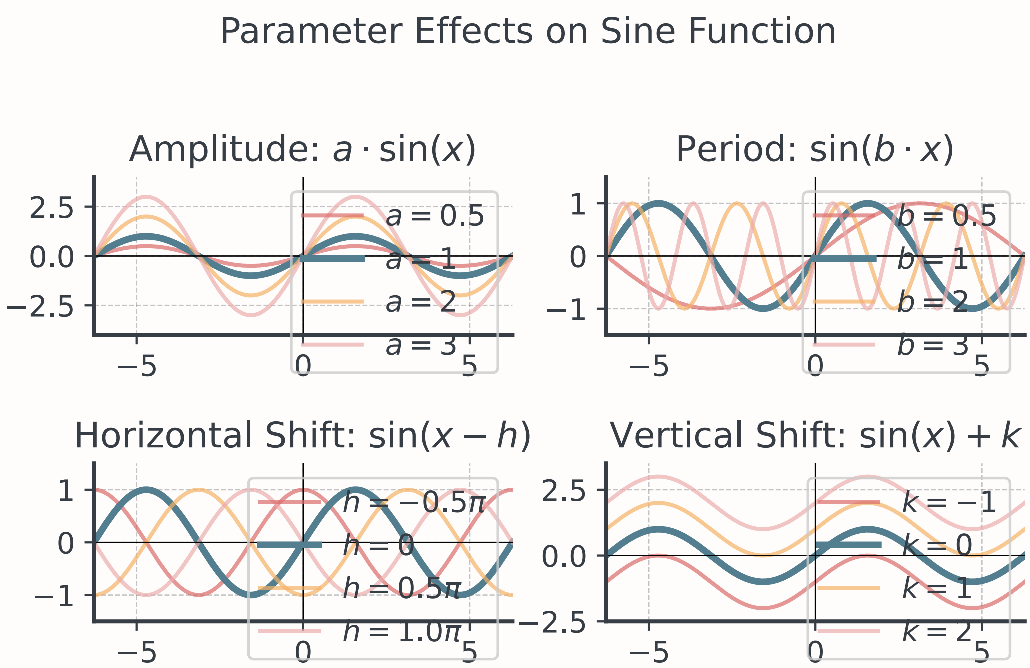
> Apply transformations in this order: horizontal shift, horizontal stretch/compress, vertical stretch/compress, vertical shift.

Transformations of Sine and Cosine II



Spot the Error: Trigonometry Mistakes

Can you identify the errors? Work with your neighbor

Time allocation: 5 minutes to find errors, 5 minutes to discuss

Student work:

1. "Since sin(30°) = 0.5, then sin(60°) = 1"
2. "tan(90°) = sin(90°)/cos(90°) = 1/0 = ∞"
3. "The period of sin(3x) is 6π"
4. "$\cos^2(x) + \sin^2(x) = 1$ only when x = 0"

Reflect

Quickly think about these questions

- How are sine and cosine related to the unit circle?
- Why do we use radians instead of degrees in calculus?
- Think of real-world examples of periodic behavior
- Can you name three phenomena that oscillate?

Break - 10 Minutes

Real-World Applications

Sound Waves I

Music is trigonometry

A pure musical tone: $y = A\sin(2\pi f t)$

- A = amplitude (volume)
- f = frequency (pitch)
- t = time

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Question: What happens if we increase the frequency?

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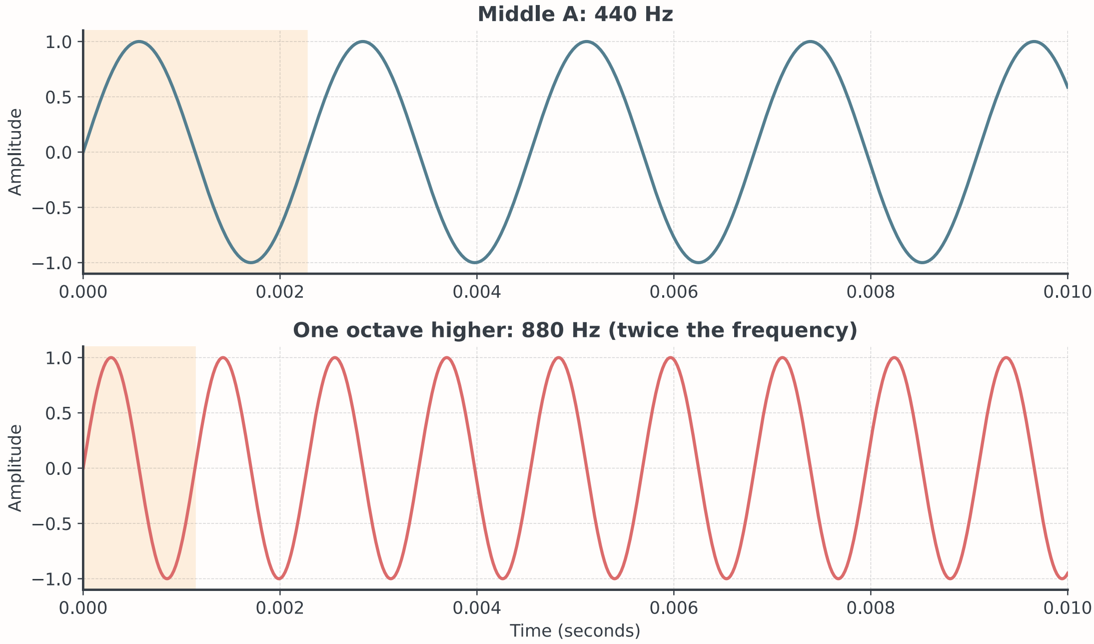
Example: Middle A (440 Hz)

$$y = \sin(2\pi \cdot 440 \cdot t) = \sin(880\pi t)$$

Sound Waves II

Comparing frequencies



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> ℹ Note
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> Notice: Doubling the frequency halves the period! The 880 Hz wave completes two cycles in the same time as 440 Hz completes one.

Seasonal Patterns I

Temperature variation

Average daily temperature in many locations:

$$T(d) = A\sin\left(\frac{2\pi}{365}(d - C)\right) + T_{avg}$$

where:

- d = day of year
- A = amplitude (half the difference between summer/winter)
- T_{avg} = average yearly temperature
- C = phase shift (adjusts when the peak occurs - typically 80-110 days)

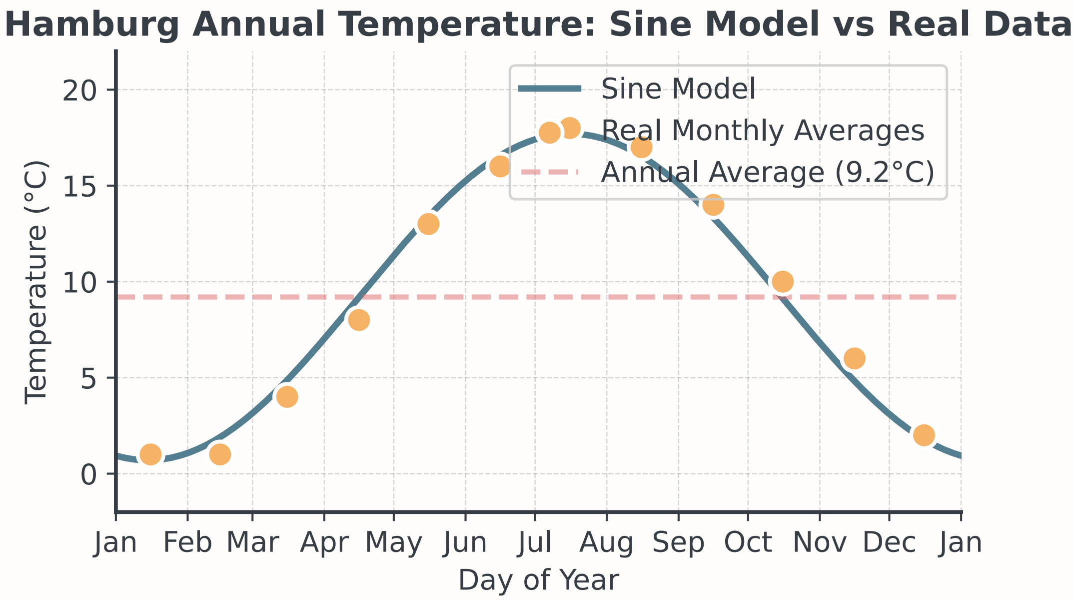
Seasonal Patterns II

Hamburg's temperature model

Using real Hamburg climate data (Weather Spark):

- Average annual temperature: $T_{avg} = 9.2°C$
- Warmest month (July): ~18°C
- Coldest months (Jan/Feb): ~1°C
- Amplitude: $A = 8.5°C$

$$T(d) = 8.5\sin\left(\frac{2\pi}{365}(d - 105)\right) + 9.2$$



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> Tip
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> The sine function provides an great fit to Hamburg's real climate data!

Guided Practice - 35 Minutes

Task 1: Analyzing Function Properties

Work alone for 5 minutes, then discuss for 3 minutes

For $y = 3\sin(2x) - 1$, find:

a) Amplitude
b) Period
c) Vertical shift
d) Range

. . .

Task 2: Tidal Heights

Work alone for 5 minutes, then discuss for 3 minutes

The water depth in a harbor varies with the tides. At high tide, the water is 12 meters deep. At low tide, it is 4 meters deep. High tide occurs at noon, and the tide cycle repeats every 12 hours.

Write a function d(t) for the water depth t hours after noon.

Hint: What is the average depth? What is the amplitude?

. . .

Task 3: Matching Graphs

Work in pairs for 5 minutes, then discuss for 3 minutes

Match each equation to its description:

Equations:

A) $y = 2\sin(x)$

B) $y = \sin(2x)$

C) $y = \sin(x) + 2$

D) $y = \sin(x - \frac{\pi}{2})$

Descriptions:

1. Is shifted up 2 units
2. Has amplitude 2
3. Completes two cycles in 2π
4. Looks like the cosine function

. . .

Task 4: Blood Pressure Modeling

Work alone for 7 minutes, then discuss for 4 minutes

A person's blood pressure oscillates with each heartbeat. Suppose a person has: a maximum pressure (systolic): 120 mmHg, minimum pressure (diastolic): 80 mmHg and a heart rate: 72 beats per minute.

Questions:

a) What is the amplitude of the blood pressure oscillation?
b) What is the period in minutes?
c) Write a function P(t) for blood pressure at time t minutes, assuming pressure starts at maximum.

. . .

Task 5: Ferris Wheel Challenge

Work alone for 5 minutes, then discuss for 3 minutes

A Ferris wheel with radius 20 meters completes one rotation every 4 minutes. The bottom of the wheel is 2 meters above ground. Write a function for the height of a rider at time t (in minutes), starting at the bottom.

. . .

Hints to consider:

- What is the center height of the wheel?
- What is the amplitude of the up-and-down motion?
- What is the period of rotation?
- Which function starts at the bottom: sine or cosine?

. . .

Coffee Break - 15 Minutes

Inverse Trigonometric Functions

Brief Addition

Going backwards

Sometimes we need to find the angle:

- If $\sin(\theta) = 0.5$, what is θ?
- Answer: $\theta = \arcsin(0.5) = \pi/6$ (or 30°)

. . .

Question: But wait! Doesn't sin(150°) also equal 0.5?

. . .

Yes! That's why we need restrictions…

Restricted Inverse

The inverse functions

- arcsin(x) or $\sin^{-1}(x)$: gives angle whose sine is x
- arccos(x) or $\cos^{-1}(x)$: gives angle whose cosine is x
- arctan(x) or $\tan^{-1}(x)$: gives angle whose tangent is x

. . .

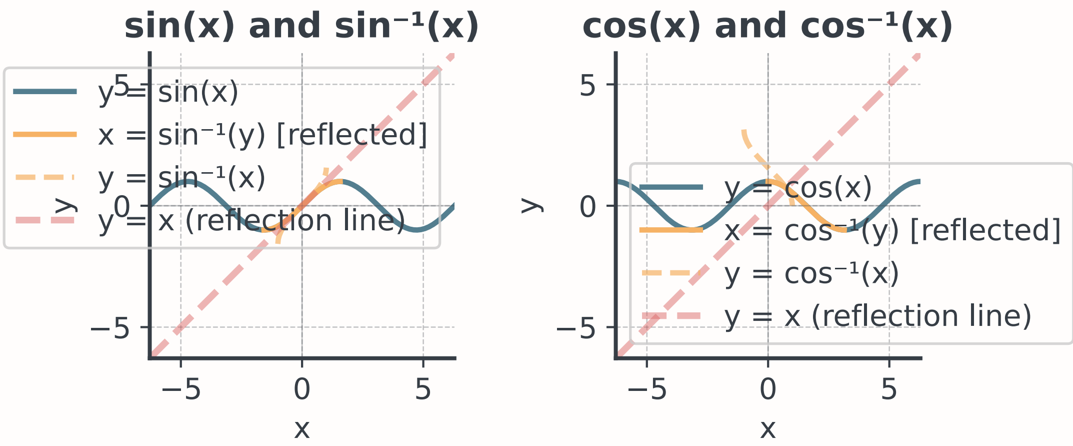
⚠ Warning

To make inverses work, we restrict the output ranges (also called principal values):

- arcsin: $[-\pi/2, \pi/2]$ (from –90° to 90°)
- arccos: $[0, \pi]$ (from 0° to 180°)
- arctan: $(-\pi/2, \pi/2)$ (from –90° to 90°, not including endpoints)

Visualizing Inverse Relationships

How sine and its inverse relate



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> Tip
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> Inverse functions are reflections across the line y = x.

Challenge Problem

Combining Waves

Work individually for 8 minutes, then discuss for 4 minutes

Consider two sound waves where the combined wave is: $y = y_1 + y_2$.

- Wave 1: $y_1 = 2\sin(3x)$
- Wave 2: $y_2 = \sin(3x + \pi)$

a) Use the fact that $\sin(x + \pi) = -\sin(x)$ to simplify y in terms of a single sine function.
b) What is the amplitude and period of the combined wave?
c) What happens if Wave 2 had amplitude 2: $y_2 = 2\sin(3x + \pi) = -2\sin(3x)$?

Key Concepts

Summary

You've learned

- Angle measurement in degrees and radians
- The unit circle and its significance
- Sine, cosine, and tangent functions
- Graphing and transformations
- Real-world periodic phenomena

Final Assessment

5 minutes - Individual work

Quick Check:

1. Convert 45° to radians
2. What is the period of y = sin(4x)?
3. What is the amplitude of y = −3cos(x) + 2?

Looking Ahead

Next Session Preview

New Function Types

- Rational functions: $f(x) = \frac{p(x)}{q(x)}$
- Radical functions beyond square root
- Reciprocal transformations

Key Concepts

- Asymptotic behavior
- Domain restrictions
- Holes vs. asymptotes
- End behavior analysis

. . .

> Tip
>
> Complete Tasks 04-04: Practice with angles, exact values, graphing, and real-world applications

Final Thought

Why Trigonometry?

Trigonometry is everywhere

From your heartbeat to the tides, from music to earthquakes, trigonometry describes the rhythms of our world.

. . .

Your phone's GPS? Triangulation with satellites Weather prediction? Modeling atmospheric waves Computer graphics? Rotation matrices Medical imaging? Fourier transforms