Session 04-02 - Power Functions & Roots

Section 04: Advanced Functions

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Entry Quiz - 10 Minutes

Review from Session 04-01

Work individually for 5 minutes, then we discuss

1. Determine the end behavior of $P(x) = -3x^4 + 2x^2 - 7$
2. Given $Q(x) = 2(x-1)^3(x+2)$, identify all zeros and their multiplicities and describe what they mean
3. If a polynomial has degree 5, what is the maximal number of turning points it can have?
4. Sketch $(x-2)(x+1)^2$ on a number line

Homework Discussion - 15 Minutes

Your questions from Tasks 04-01

Focus on polynomial applications and factoring

- Challenges with the Rational Root Theorem
- Sketching polynomials from factored form
- Interpreting multiplicity in graphs
- Business context questions

...

> **ℹ Note**
>
> Power functions will help us understand the individual components of polynomials!

Learning Objectives

Today's Goals

By the end of this session, you will be able to:

- Understand power functions with integer, fractional, and negative exponents
- Determine domains of root functions and fractional powers

- Sketch power function graphs without a calculator
- Compare growth rates of different power functions
- Apply power functions to economic models

Power Functions with Integer Exponents

Definition and Basic Forms

The building blocks of polynomials

A power function has the form:

$$f(x) = ax^n$$

where $a \neq 0$ is a constant and n is a real number

. . .

- Integer powers: x^2, x^3, x^{-1}, x^{-2}
- Root functions: $\sqrt{x}$, $\sqrt[3]{x}$
- Fractional powers: $x^{2/3}, x^{3/2}$

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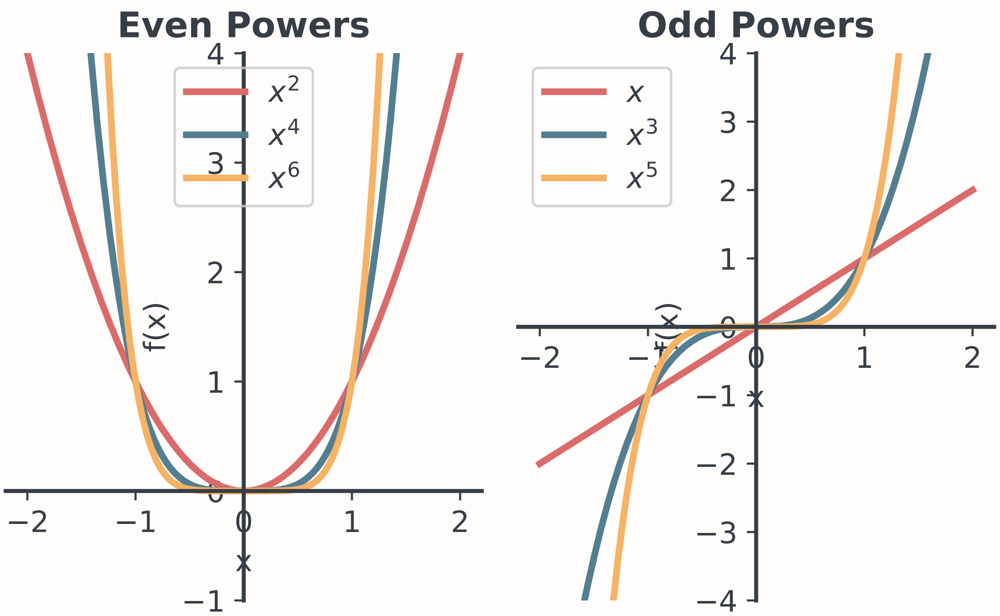
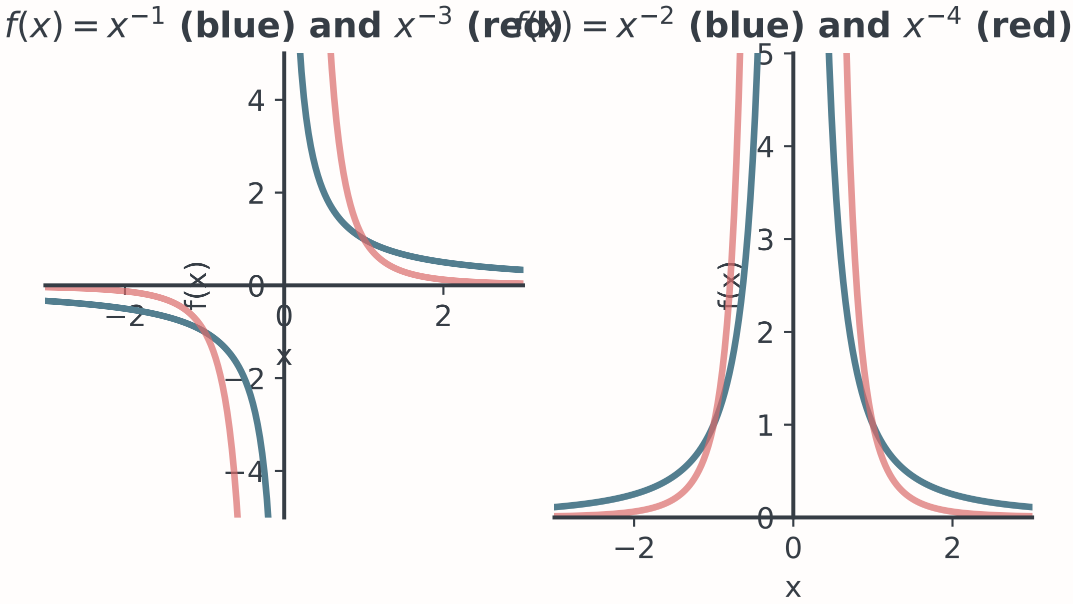
> ℹ Note
>
> Power functions are simpler than polynomials! But they reveal fundamental behaviors that help us understand all functions.

Comparing even and odd powers



Negative Integer Powers (Reciprocal)

For $n > 0$: $f(x) = x^{-n} = \frac{1}{x^n}$, not defined at $\boldsymbol{x = 0}$



Think-Pair-Share

2 minutes individual, 3 minutes pairs, 2 minutes class discussion

Compare and contrast:

1. $f(x) = x^2$ and $g(x) = x^{-2}$
2. $f(x) = x^3$ and $g(x) = x^{-3}$

Consider:

- Domain and range
- Symmetry
- Behavior near $x = 0$
- Behavior as $x \to \pm\infty$

. . .

Discussion Points

- Positive powers have $x = 0$ in domain; negative powers don't
- Both maintain same type of symmetry (even/odd)
- Reciprocal relationship: large becomes small, small becomes large
- Asymptotic behavior reverses

Root Functions

Functions with fractional exponents

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<>:19: SyntaxWarning: invalid escape sequence '\s'
<>:20: SyntaxWarning: invalid escape sequence '\s'
<>:21: SyntaxWarning: invalid escape sequence '\s'
<>:38: SyntaxWarning: invalid escape sequence '\s'
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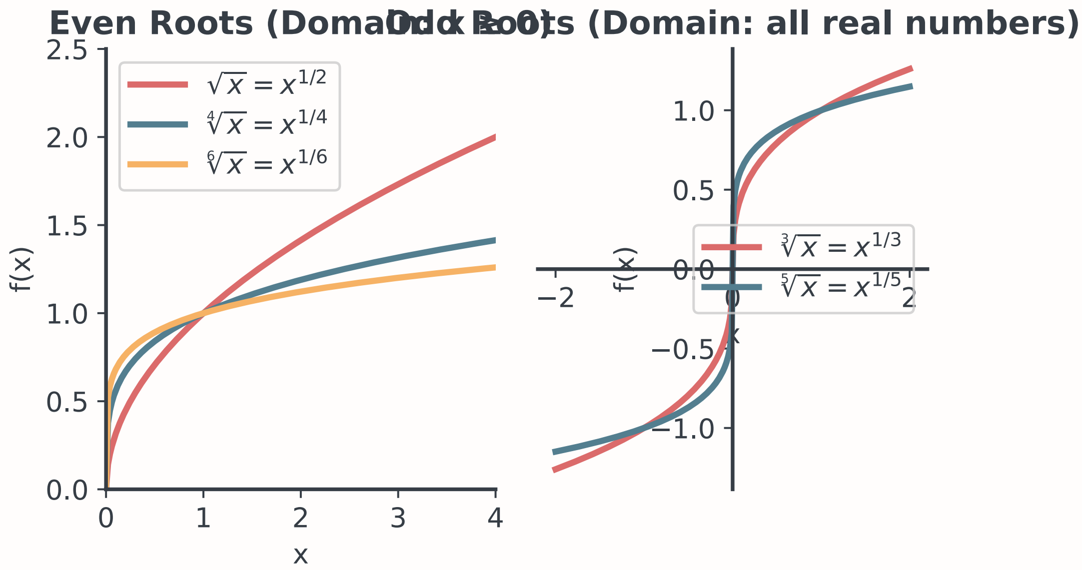
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_66640/1185525214.py:19: SyntaxWarning: invalid escape sequence
'\s'
  axes[0].plot(x_even, np.sqrt(x_even), label='$\sqrt{x} = x^{1/2}$',
linewidth=2.5)
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_66640/1185525214.py:20: SyntaxWarning: invalid escape sequence
'\s'
  axes[0].plot(x_even, x_even**(1/4), label='$\sqrt[4]{x} = x^{1/4}$',
linewidth=2.5)
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_66640/1185525214.py:21: SyntaxWarning: invalid escape sequence
'\s'
  axes[0].plot(x_even, x_even**(1/6), label='$\sqrt[6]{x} = x^{1/6}$',
linewidth=2.5)
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_66640/1185525214.py:38: SyntaxWarning: invalid escape sequence
'\s'
  axes[1].plot(x_odd, np.cbrt(x_odd), label='$\sqrt[3]{x} = x^{1/3}$',
linewidth=2.5)
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_66640/1185525214.py:39: SyntaxWarning: invalid escape sequence
'\s'
  axes[1].plot(x_odd, np.sign(x_odd) * np.abs(x_odd)**(1/5), label='$
\sqrt[5]{x} = x^{1/5}$', linewidth=2.5)
```



Domain Restrictions

Critical concept for root functions

Even roots

like $\sqrt{x}$, $\sqrt[4]{x}$:

- Domain: $x \geq 0$ only
- Why? Even roots of negative numbers aren't real
- $\sqrt{x^2} = |x|$ (always non-negative)
- Range: $y \geq 0$

Odd roots

like $\sqrt[3]{x}$, $\sqrt[5]{x}$:

- Domain: all real numbers
- Can take cube root of negative numbers
- Range: all real numbers

Fractional Powers

General Fractional Exponents

Combining powers and roots

For $f(x) = x^{m/n}$ where m, n are integers, $n > 0$:

$$x^{m/n} = \left(x^{1/n}\right)^m = \sqrt[n]{x^m}$$

...

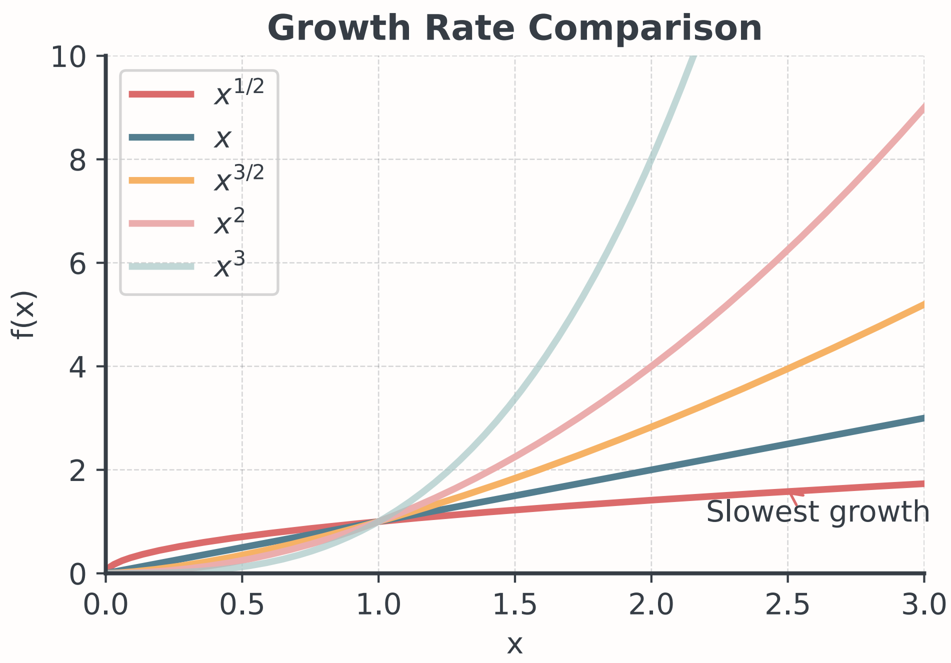
Domain depends on n:

- If n is odd: all real numbers (usually)
- If n is even: $x \geq 0$ required

...

> **Note**
>
> Example: $f(x) = x^{3/2} = \sqrt{x^3} = \left(\sqrt{x}\right)^3$
>
> - Domain: $x \geq 0$ (because of square root), grows faster than $\sqrt{x}$ but slower than x^2

Comparing Growth Rates



Break - 10 Minutes

Economic Applications

Economies of Scale

Cost functions with fractional powers

Many production processes exhibit economies of scale:

$$C(x) = 500 + 50x^{0.7}$$

where x is production quantity (thousands)

...

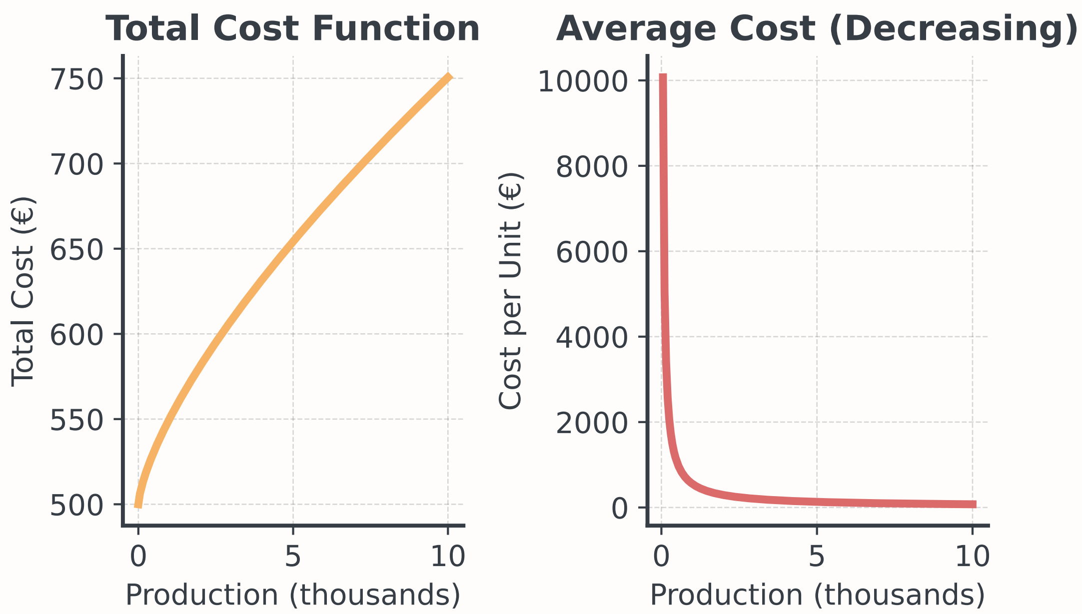
Question: Any idea why $x^{0.7}$?

. . .

Exponent < 1 means cost grows slower than production!

Economies of Scale II

```
/var/folders/_5/jkkjxxdd5f1955l380dky7n80000gn/T/
ipykernel_66640/2782387132.py:9: RuntimeWarning: divide by zero encountered
in divide
  C_average = C_total / x  # Now both arrays have same length
```



Allometric Growth I

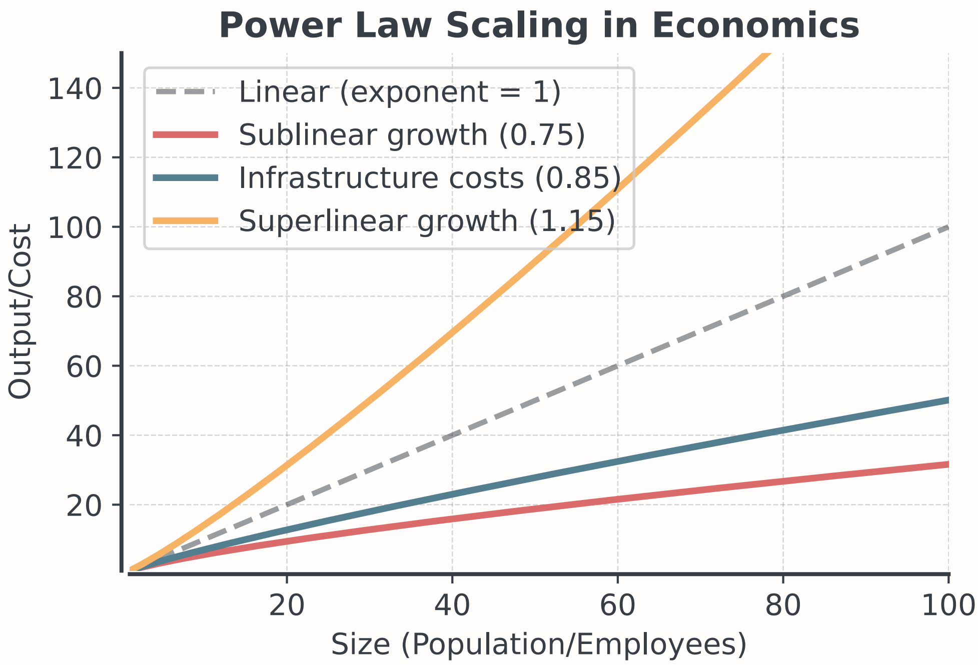
Biological and economic scaling

Many relationships follow power laws:

- Biology: Brain mass $\propto$ (Body mass)$^{0.75}$
- Economics:
  - City infrastructure costs $\propto$ (Population)$^{0.85}$
  - Company revenue $\propto$ (Number of employees)$^{1.15}$

. . .

> Tip
>
> Have you ever seen $\propto$ before? That's the proportionality symbol ($\propto$), which means "is proportional to" in mathematics.



Surface Area and Volume

Geometric Power Relationships

Why packaging costs don't scale linearly

Example: Doubling box dimensions

- Surface area increases by factor of $2^2 = 4$
- Volume increases by factor of $2^3 = 8$
- Material cost (surface) vs. capacity (volume)

. . .

> Tip
>
> This explains why larger packages have lower cost per unit volume. The discount has a mathematical basis!

Graphing Techniques

Sketching Power Functions

A systematic approach

Steps to sketch $f(x) = ax^n$ or $f(x) = ax^{m/n}$:

1.

Determine domain

- Negative exponents: exclude $x = 0$
- Even roots: require $x \geq 0$

2.

Find key points

- Always passes through $(1, a)$ if in domain
- Check $(0, 0)$ if applicable

3.

Analyze end behavior

- Positive exponents: consider even/odd
- Negative exponents: approach axes

4.

Check symmetry

- Even exponents: y-axis symmetry
- Odd exponents: origin symmetry

Practice: Sketch Without Calculator

Work together

Sketch: $f(x) = 2x^{2/3}$

Practice Domain and Range Analysis

Work together and discuss

For each function, determine the domain and range, then sketch a rough graph:

a) $f(x) = 3x^{1/4}$

b) $g(x) = -2x^{-1}$

c) $h(x) = x^{3/5}$

d) $p(x) = 4 - x^{1/2}$

Coffee Break - 15 Minutes

Combining Power Functions

Sums and Products

Building complex models

Real-world phenomena often combine power functions:

Total Cost with Multiple Effects:

$$C(x) = 1000x^{0.5} + 50x + 0.1x^2$$

- $x^{0.5}$: Setup costs (economies of scale)
- x: Linear variable costs
- x^2: Capacity constraints (diseconomies)

Guided Practice - 25 Minutes

Individual Exercise Block I

Work alone for 5 minutes, then discuss for 5 minutes

Problem 1: A company's profit function combines multiple effects:

$$P(x) = -2x^3 + 15x^2 + 100\sqrt{x} - 500$$

where x is production level (hundreds of units), $x > 0$

a) Identify each term's economic interpretation
b) Calculate $P(4)$ and $P(9)$
c) Which term dominates for large x? What does this tell management?

Individual Exercise Block II

Work alone for 5 minutes, then discuss for 5 minutes

Problem 2: Compare growth rates for large values:

a) Which grows faster: $f(x) = x^{1/2}$ or $g(x) = x^{1/3}$?
b) Which grows faster: $f(x) = x^{3/2}$ or $g(x) = x^2$?
c) Order from slowest to fastest growth: $x^{1/2}, x, x^{3/2}, x^2$

Individual Exercise Block III

Work alone for 5 minutes, then discuss for 5 minutes

Problem 3: A technology company's average cost per unit is:

$$AC(x) = \frac{50000}{x} + 100 + 0.01x$$

where x is units produced.

a) Identify the power function in each term
b) What happens to average cost as $x \to 0^+$?
c) What happens to average cost as $x \to \infty$?
d) Graph the behavior conceptually

Spot the Error

Can you find what's wrong? Work with your neighbor

Time allocation: 5 minutes to find errors, 5 minutes to discuss

Student work:

1. "$\sqrt{x^2} = x$ for all x"
2. "The function $f(x) = x^{-1/2}$ has domain $x > 0$"
3. "Since $x^{2/3} = \sqrt[3]{x^2}$, the domain is $x \geq 0$"
4. "$x^{1.5}$ grows faster than x^2 because 1.5 is complicated"

. . .

Tip

Not everything has to be wrong!

Wrap-Up

Key Takeaways

Today's essential concepts

- Power functions are the building blocks of polynomials
- Domain restrictions come from mathematical necessity
- Growth rate comparisons guide long-term planning
- Fractional powers create realistic economic models

Final Assessment

5 minutes - Individual work

A manufacturing company's cost per unit follows:

$$C(x) = 10000x^{-0.5} + 50 + 2x^{0.5}$$

where x is the number of units produced (in thousands).

1. What is the domain of this function in the business context?
2. Identify each term's economic meaning
3. What happens to cost per unit as production increases dramatically?
4. Which term represents economies of scale?

Next Session Preview

Session 04-03: Exponential Functions

Moving from power to exponential growth

- Exponential growth and decay fundamentals
- The natural exponential e^x and its properties

- Compound interest and continuous growth
- Population models and doubling time
- Exponential vs. power function growth (critical comparison!)
- Half-life and decay applications

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> **Note**
>
> Complete Tasks 04-02!

Thank You!

Appendix: Sketch without calculator