Session 03-05 - Composition, Inverses & Advanced Graphing

Section 03: Functions as Business Models

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Entry Quiz - 10 Minutes

Review from Session 03-04

Work individually, then exchange with your neighbor for peer review

1. Given $f(x) = 2x^2 - 8x + 6$:
   a) Write the function shifted left 3 units
   b) Write the function stretched vertically by factor 1.5
2. A cost function $C(x) = 100 + 5x$ increases fixed costs by €50 and variable costs by 20%. Write the new function.
3. Identify the transformations needed to go from $f(x) = x^2$ to $g(x) = -2(x+1)^2 + 4$
4. If a profit function's graph shifts right by 10 units, what does this mean for the business?

Homework Discussion - 20 Minutes

Your questions from Tasks 03-04

Focus on transformation interpretations

- Problem 4: Seasonal business model
  - How did season reversal affect profits?
- Problem 5: Graph interpretation
  - Identifying cost-effective production ranges
- Problem 7: Multi-location analysis
  - Which transformations most impacted profitability?

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> **Note**
>
> Composition and inverses will help us model even more complex business relationships!

Function Composition

Understanding Composition

Composition models sequential processes

Definition: $(f \circ g)(x) = f(g(x))$

- Read as "f composed with g"
- Apply g first, then f
- Output of g becomes input of f
- Order matters! $(f \circ g) \neq (g \circ f)$ usually
- Business meaning: Multi-step processes

Composition Example: Supply Chain

Imagine a company manufacturing products from raw materials.

1. Raw materials to components: $g(x) = 2x + 10$ (cost in €)
2. Components to products: $f(y) = 3y + 50$ (cost in €)

- $(f \circ g)(x) = f(g(x)) = f(2x + 10)$
- $= 3(2x + 10) + 50$
- $= 6x + 30 + 50$
- $= 6x + 80$
- For 10 units raw material: Cost = €140

Supply Chain Visualization

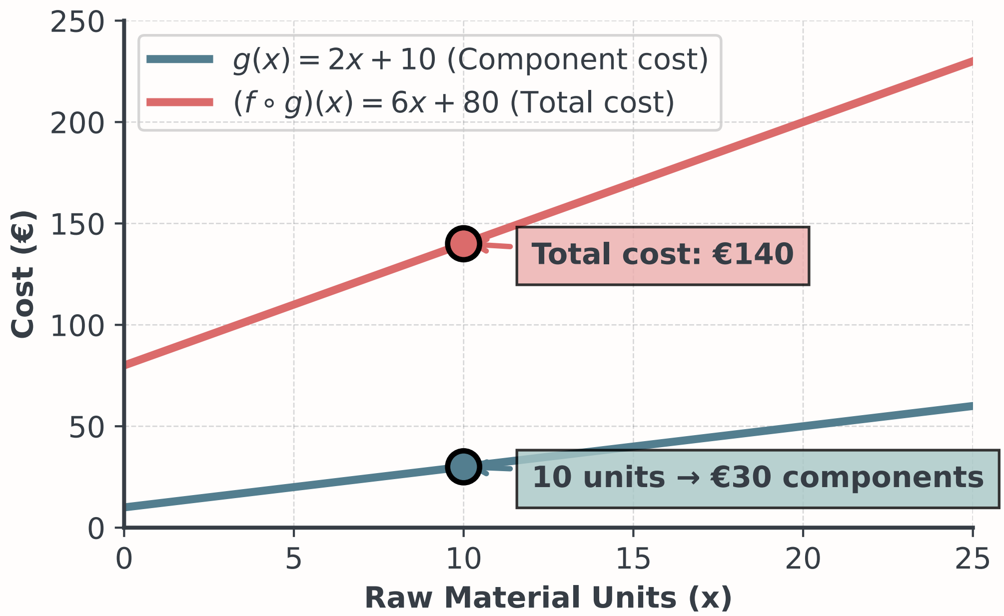

Figure 1: Function Composition: Supply Chain Cost Analysis

Domain Considerations

Domain of composition can be restricted

For $(f \circ g)(x)$:

1. Start with domain of g
2. Find range of g
3. Intersect with domain of f
4. Track back to valid x values

. . .

> Tip
>
> This is not too complicated, right? Here we just need to be careful.

Example: Weight-Based Dosage and Safe Maximum

Calculate dosage based on body weight, then check safety limit

- Dosage from weight: $g(x) = 5x$ mg (5mg per kg body weight)
- Safe processing: $f(x) = \sqrt{500 - x}$ (requires total dose ≤ 500 mg)
- Composition requires: $5x \leq 500$, so $x \leq 100$ kg body weight

. . .

> Warning
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> Therefore, we cannot process a patient weighing more than 100kg, as this would be unsafe!

Quick Practice - 10 Minutes

Work individually, then we discuss

A food delivery service has the following cost structure:

- Restaurant to warehouse: $g(x) = 1.5x + 5$ (€ per order, where x is number of orders)
- Warehouse to customer: $f(y) = 2y + 8$ (€ delivery cost)

1. Find the composition $(f \circ g)(x)$ representing total delivery cost.

2. Calculate the total cost for 20 orders.

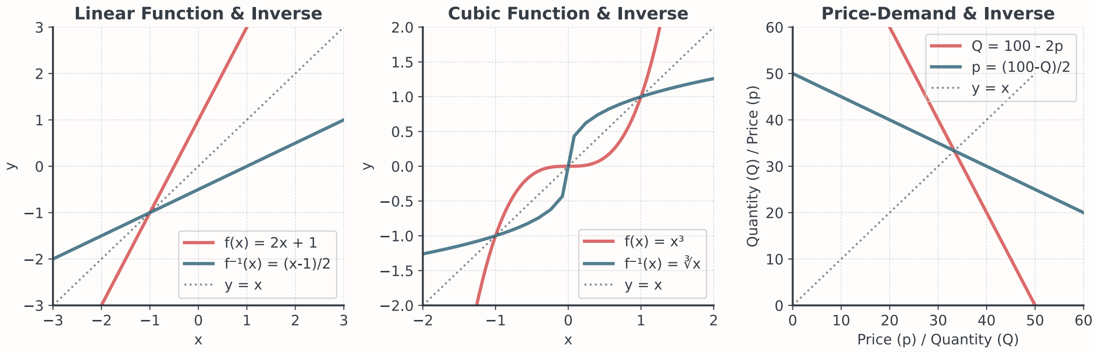
3. If the domain of g is $[0, 100]$ orders, what is the range of g?

Break - 10 Minutes

Inverse Functions

What is an Inverse Function?

An inverse function reverses the original function



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💡 Tip

If $f(a) = b$, then $f^{-1}(b) = a$. The inverse "undoes" what the original function does.

Testing for Invertibility

A function has an inverse if it's one-to-one

- Each output comes from exactly one input
- Horizontal line test: Each horizontal line hits graph at most once
- For continuous functions: Always increasing or always decreasing

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Question: Which of the following two fulfill the one-to-one condition?

- $f(x) = 2x + 5$
- $g(x) = x^2$

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💡 Tip

Just make sure, you don't use the vertical line test here!

Finding Inverse Functions

Step-by-step process

1. Replace $f(x)$ with y

2. Swap x and y
3. Solve for y
4. Replace y with $f^{-1}(x)$
5. Verify domain and range

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> Tip
>
> Looks complicated? It is actually rather easy!

Example: Simple Linear Function

Find the inverse of $\boldsymbol{f(x) = 3x + 6}$

1. $y = 3x + 6$
2. $x = 3y + 6$ (swap x and y)
3. $3y = x - 6$ (solve for y)
4. $y = \frac{x-6}{3}$
5. Inverse: $f^{-1}(x) = \frac{x-6}{3}$

...

> Tip
>
> Verify: $f(f^{-1}(x)) = f\left(\frac{x-6}{3}\right) = 3 \cdot \frac{x-6}{3} + 6 = x - 6 + 6 = x$ ✓

Visualizing the Inverse

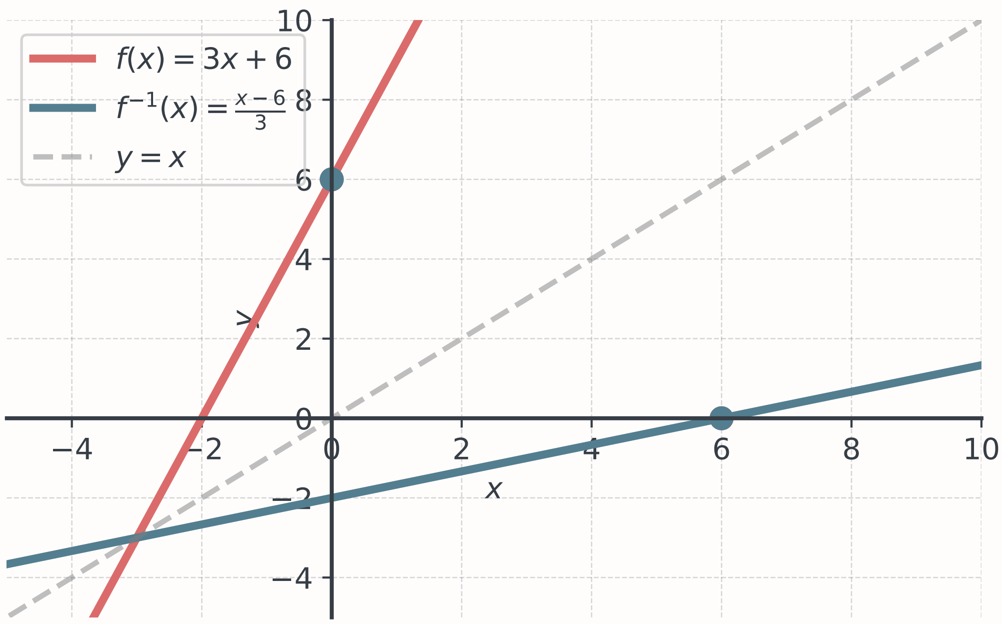

Figure 2: Function $f(x) = 3x + 6$ and its inverse $f^{-1}(x) = \frac{x-6}{3}$ reflected over $y = x$

Example: Price-Demand Inverse

Demand function: $\mathbf{Q = 1000 - 20p}$

1. $y = 1000 - 20p$
2. $x = 1000 - 20p$ (swap)
3. $x - 1000 = -20p$
4. $p = 50 - 0.05x$
5. Inverse: $p(Q) = 50 - 0.05Q$

...

> Tip
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> What does this mean? It gives the price needed for specific quantity!

Visualizing Price-Demand Relationship

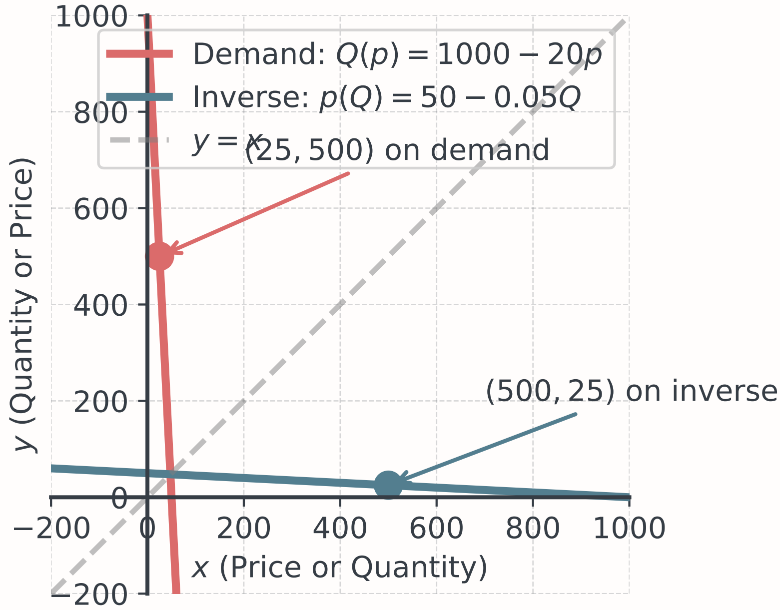

Figure 3: Demand function and its inverse reflected over the line $y = x$.

Advanced Graphing Techniques

Graphing Composed Functions

Build graphs step by step

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Strategy for graphing $(f \circ g)(x)$:

1. Graph $g(x)$ first
2. For key points, find $g(x)$ values

3. Apply f to those values
4. Plot resulting points
5. Consider domain restrictions

Graphing Inverse Functions

Use reflection property

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To graph $f^{-1}(x)$ from $f(x)$:

1. Plot original function $f(x)$
2. Draw line $y = x$
3. Reflect $f(x)$ over this line
4. Key points: (a, b) on $f \longrightarrow (b, a)$ on f^{-1}
5. Domain of f = Range of f^{-1}
6. Range of f = Domain of f^{-1}

Guided Practice - 25 Minutes

Individual Exercise Block

Work alone for 15 minutes, then discuss for 10 minutes

1. Given $f(x) = 2x + 3$ and $g(x) = x^2 - 1$:
   a) Find $(f \circ g)(x)$ and evaluate $(f \circ g)(2)$
   b) Find $(g \circ f)(x)$ and evaluate $(g \circ f)(2)$
2. A company converts raw materials through two stages:
   - Stage 1: $C_1(x) = 50x + 200$ (process cost in €)
   - Stage 2: $C_2(y) = 2y + 500$ (assembly cost in €)
   - Find the total cost function and cost of 100 units of material.
3. Find the inverse of $f(x) = \frac{2x+3}{5}$ and verify your answer.

Coffee Break - 15 Minutes

Business Process Modeling

Multi-Step Processes

Complex business operations as function chains

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Question: Does you have any example that involves multiple stages?

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Examples of composition in business:

- Manufacturing: Raw materials $\longrightarrow$ Components $\longrightarrow$ Products

- Finance: Local currency $\longrightarrow$ Foreign currency $\longrightarrow$ Investment
- Retail: Wholesale price $\longrightarrow$ Retail price $\longrightarrow$ After-tax price
- Marketing: Leads $\longrightarrow$ Conversions $\longrightarrow$ Revenue

Supply Chain Example

Three-Stage Production from Cost to Price

1. Supplier: Raw materials at $p_r = 10 + 0.5x$ (€ per kg)
2. Processor: Converts at 80% efficiency, adds €20/kg
3. Retailer: Marks up 50%, adds €15 flat fee

. . .

Let's build the final price function together:

- After supplier: Cost = $x(10 + 0.5x) = 10x + 0.5x^2$
- After processor: Cost = $0.8(10x + 0.5x^2) + 20x$
- After retailer: Price = $1.5[(10x + 0.5x^2) + 20x] + 15$

Inverse Functions in Business

Common business inverses

- Demand $\leftrightarrow$ Price: Know one, find the other
- Cost $\leftrightarrow$ Quantity: Determine production from budget
- Profit $\leftrightarrow$ Sales: Find required sales for target profit
- Break-even analysis: Reverse engineer requirements

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> Tip
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> Also, it could just be a task you are given.

Problem-Solving - 30 Minutes

International Business Model I

The Scenario: Global E-Commerce Platform

An e-commerce company operates internationally with these functions:

Pricing Model:

- Base price in USD: $P(x) = 50 + 0.8x$ where x is quantity
- EUR conversion: $E(p) = 0.85p$
- UK markup: $U(p) = 1.2p + 10$

International Business Model II

The Scenario: Global E-Commerce Platform

Shipping Costs:

- Weight calculation: $W(x) = 0.5x + 2$ (kg)
- Shipping cost: $S(w) = 15w + 25$ (€)

Customer Demand:

- At total price T: $D(T) = 5000 - 20T$ units per month

Your Tasks

If you like, you can work in groups

1. Find the total price function for EU customers (product + shipping) as a function of quantity x.
2. Find the inverse of the demand function. What does it represent?
3. Create a composite function that gives monthly demand based on quantity ordered.
4. If the company wants exactly 2000 units demanded per month, what should the quantity per order be?
5. The company can only process orders where total price leads to positive demand. Find the maximum viable order quantity.

Section 03 Synthesis

Your Function Toolkit

Complete arsenal for business modeling

- Function basics: Notation, domain, range
- Linear functions: Constant rates, equilibrium
- Quadratic functions: Optimization, vertex
- Transformations: Adapting to change
- Composition: Multi-step processes
- Inverses: Reversing calculations

Key Business Applications

From this section, you can now:

- Model costs, revenue, and profit
- Find market equilibrium
- Optimize prices and quantities
- Adapt models for new conditions
- Chain processes together
- Reverse-engineer requirements

Wrap-Up

Key Takeaways

- Composition models sequential processes
- Order matters: $(f \circ g) \neq (g \circ f)$
- Inverses reverse calculations
- One-to-one functions are invertible
- Graphical tools reveal function relationships
- Business processes often involve multiple functions

Final Assessment

5 minutes - Individual work

A retailer has:

- Cost function: $C(x) = 20x + 500$
- Price function: $P(x) = 50 - 0.5x$
- Revenue: $R(x) = x \cdot P(x)$

1. Express revenue as a function of x explicitly
2. Find the inverse of the cost function
3. If the retailer has a budget of €2,500, how many units can they stock?

Next Session Preview

Session 03-06: Mock Exam 02
Key concepts you should master

- Function fundamentals: Domain, range, notation
- Linear functions: Supply/demand, equilibrium, CVP analysis
- Quadratic functions: Vertex, optimization, profit maximization
- Transformations: Shifts, stretches, business interpretations
- Composition & Inverses: Multi-step processes, reversing calculations
- Graphical analysis: Reading and interpreting business graphs

Exam Preparation Tips

Function Problem Strategies

Systematic approaches for success

- Start with domain: Always check restrictions first
- Identify function type: Linear, quadratic, or composed?
- Business context: What does each variable represent?
- Optimization: Remember vertex formula $x = -\frac{b}{2a}$
- Verify results: Do answers make business sense?

Common Pitfalls to Avoid

Learn from typical mistakes

- Not follwing the instructions: Read the tasks carefully!
- Composition order: Remember $(f \circ g) \neq (g \circ f)$
- Units confusion: Track currency vs. quantity units
- Graph interpretation: Missing key features
- Business meaning: Failing to interpret mathematical results

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> Tip
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> Homework Assignment: Complete Tasks 03-05!