Session 03-04 - Transformations & Graphical Analysis

Section 03: Functions as Business Models

Dr. Nikolai Heinrichs & Dr. Tobias Vlćek

Entry Quiz - 10 Minutes

Review from Session 03-03

Work individually, then we compare together

1. Find the vertex of $f(x) = -2x^2 + 12x - 10$ and determine if it's a maximum or minimum.
2. A company's profit function is $P(x) = -x^2 + 80x - 1200$. Find:
   a) The quantity that maximizes profit
   b) The maximum profit
3. Convert $g(x) = x^2 - 4x + 7$ to vertex form.
4. Revenue is modeled by $R(p) = p(600 - 3p)$. What price maximizes revenue?

Homework Discussion - 20 Minutes

Sharing Solutions from Tasks 03-03

Focus on optimization strategies

- Problem 3: Theater revenue optimization
  - How did capacity constraints affect your solution?
- Problem 5: Area optimization with fencing
  - Impact of the river (no fence needed)?
- Problem 7: Multi-product optimization (if attempted)
  - Bundling vs. separate pricing insights?

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> Key Insight
>
> Optimization often involves balancing mathematical ideals with practical constraints!

Vertical Transformations

Vertical Shifts

Moving graphs up or down

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Given original function $f(x)$:

- Upward shift: $g(x) = f(x) + k$ (k > 0)
- Downward shift: $g(x) = f(x) - k$ (k > 0)
- Graph effect: Entire graph moves vertically
- Business meaning:
  - Fixed cost changes
  - Base price adjustments
  - Overhead modifications

Example: Cost Function Adjustment

Original cost: $C(x) = 5x^2 + 3x + 100$

- Rent increases by €100: $C_{new}(x) = 5x^2 + 3x + 200$
- Government subsidy of €50: $C_{new}(x) = 5x^2 + 3x + 50$

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Question: Any idea how we could graph this?

Vertical Shift Visualization

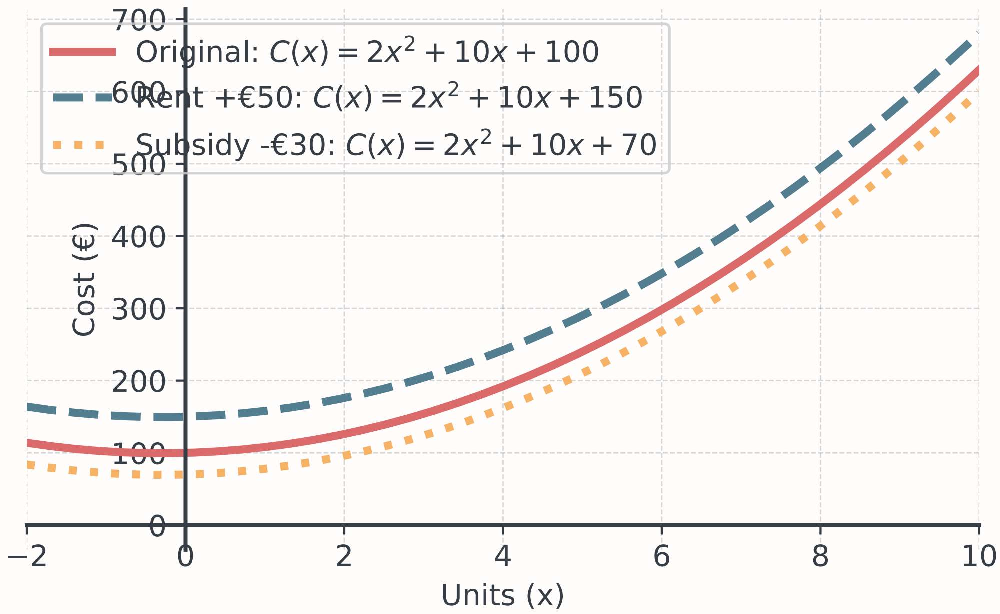

Figure 1: Vertical Shifts in Cost Functions

Same parabola shape, shifted vertically - fixed costs change, but variable cost structure remains the same!

Vertical Stretching and Compression

Changing the vertical scale

. . .

Given original function $f(x)$:

- Vertical stretch: $g(x) = a \cdot f(x)$ where $a > 1$
- Vertical compression: $g(x) = a \cdot f(x)$ where $0 < a < 1$
- Reflection: $g(x) = -f(x)$ (flip over x-axis)
- Business meaning:
  - Percentage markups/discounts
  - Tax multipliers
  - Currency conversions

Example: Revenue Scaling

Original revenue: $R(x) = 50x - 0.5x^2$

- 20% price increase across all products: $R_{new}(x) = 1.2(50x - 0.5x^2) = 60x - 0.6x^2$
- 20% price decrease across all products: $R_{new}(x) = 0.8(50x - 0.5x^2) = 40x - 0.4x^2$

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Question: Can anyone describe what happens now?

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> Tip
>
> All three functions have the same optimal quantity (50 units), but revenue scales proportionally with price - stretch up or compress down!

Vertical Stretch and Compression

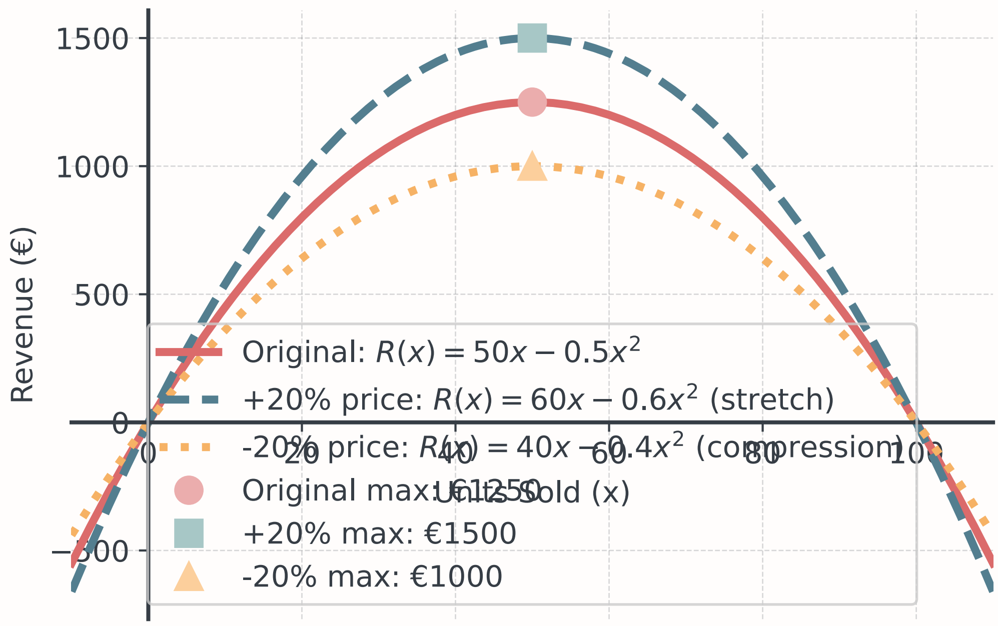

Figure 2: Vertical Stretch & Compression: Price Changes

Reflection Visualization

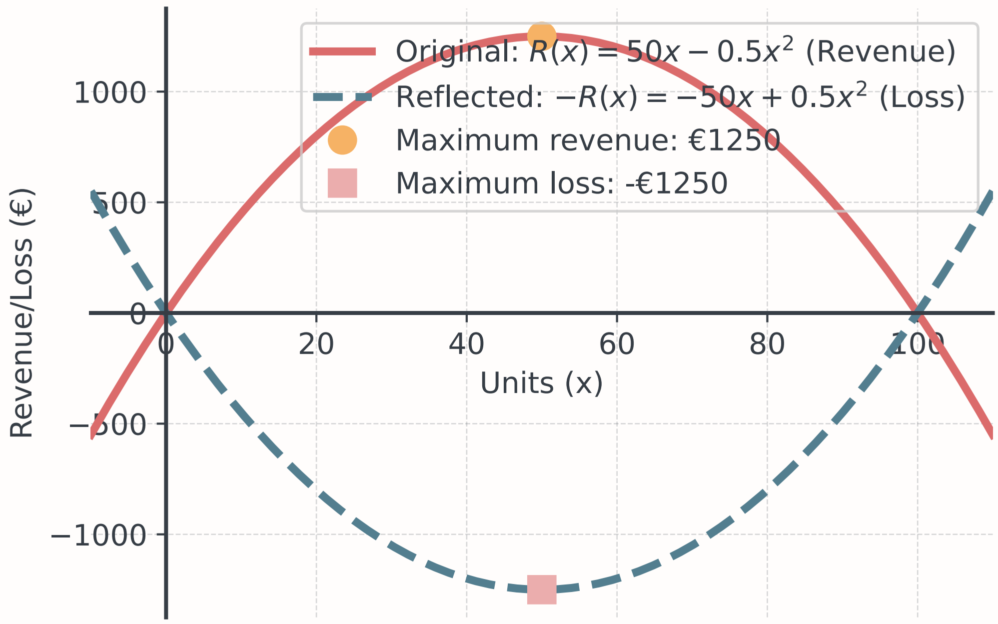

Figure 3: Reflection: Flipping Over x-axis

Quick Practice - 10 Minutes

Work individually, then we discuss

Given the original profit function: $P(x) = -x^2 + 40x - 200$

1. Write the new function if fixed costs increase by €100
2. Write the new function if a government grant reduces costs by €50
3. Write the new function if all revenues increase by 30%
4. Write the new function if all revenues decrease by 25%
5. What business scenario could $-P(x)$ represent?
6. Write the reflected function explicitly

Break - 10 Minutes

Horizontal Transformations

Horizontal Shifts

Moving graphs left or right

. . .

Given original function $f(x)$:

- Right shift: $g(x) = f(x - h)$ where $h > 0$
- Left shift: $g(x) = f(x + h)$ where $h > 0$
- Business meaning:
  - Time delays or advances
  - Market entry timing
  - Seasonal adjustments

. . .

> ⚠ Warning
>
> Counterintuitive: Minus shifts right, plus shifts left!

Example: Seasonal Demand Shift

Summer demand peaks in June (month 6): $D(t) = -(t - 6)^2 + 100$

- Unusually hot weather shifts peak to May (left shift): $D_{new}(t) = -(t - 5)^2 + 100$
- Unusually cold weather shifts peak to July (right shift): $D_{new}(t) = -(t - 7)^2 + 100$

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Question: Anyone with an idea how to graph this?

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> Tip
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> Same shape parabola, shifted horizontally - peak demand moves but pattern stays the same!

Horizontal Shift Visualization

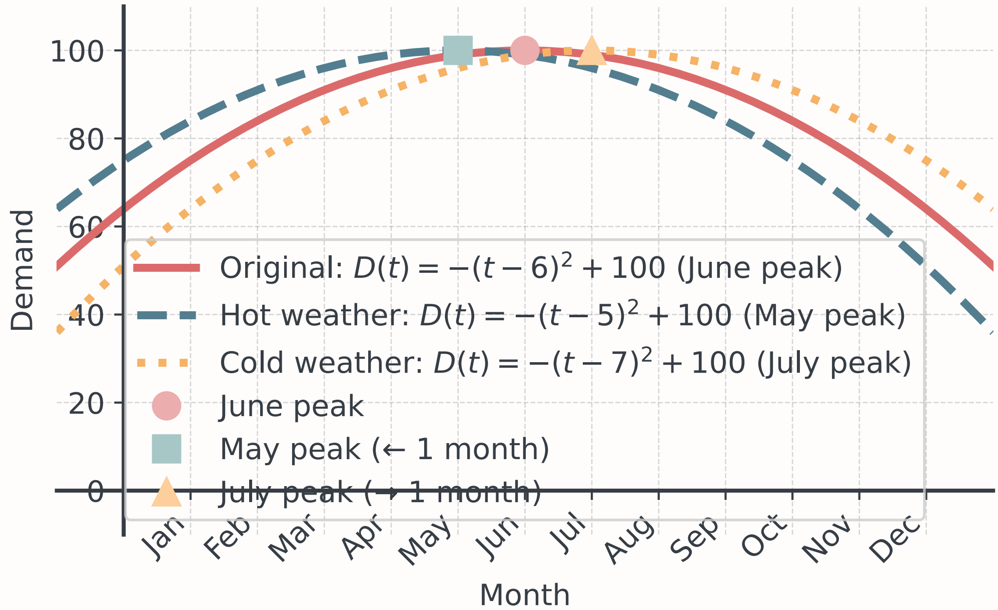

Figure 4: Horizontal Shifts: Seasonal Demand Changes

⚠ Warning

Remember: $D(t-5)$ shifts RIGHT to May, $D(t-7)$ shifts RIGHT to July - counter-intuitive notation!

Horizontal Stretch and Compression

Changing the horizontal scale

. . .

Given original function $f(x)$:

- Horizontal stretch: $g(x) = f(x/b)$ where $b > 1$
- Horizontal compression: $g(x) = f(bx)$ where $b > 1$
- Reflection: $g(x) = f(-x)$ (flip over y-axis)
- Business meaning:
  - Time scaling (quarterly to monthly)
  - Production speed changes
  - Market cycle adjustments

Example: Product Lifecycle

Original lifecycle (monthly): $L(t) = -t^2 + 8t + 1000$

- Competitor speeds up cycle (2x faster): $L_{fast}(t) = -(2t)^2 + 8(2t) = -4t^2 + 16t + 1000$

- Extended warranty slows cycle (2x slower): $L_{slow}(t) = -(t/2)^2 + 8(t/2) = -0.25t^2 + 4t + 1000$

...

Question: What happens to the lifecycle duration?

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> Tip
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> Horizontal compression (faster) → narrower curve, earlier peak.
>
> Horizontal stretch (slower) → wider curve, later peak!

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> Warning
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> Counterintuitive again: $f(2t)$ compresses (faster), $f(t/2)$ stretches (slower)!

Horizontal Stretch/Compression

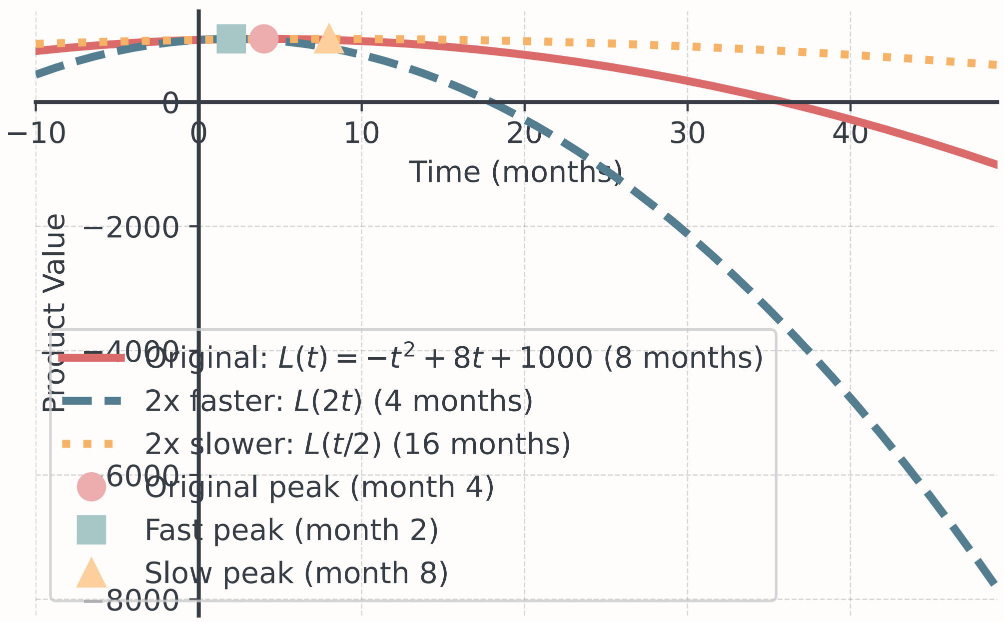

Figure 5: Horizontal Scaling: Product Lifecycle Changes

Combining Transformations

Order of Operations

Apply transformations systematically

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Standard order for $g(x) = a \cdot f(b(x-h)) + k$:

1. Horizontal shift by h
2. Horizontal stretch/compress by factor b
3. Vertical stretch/compress by factor a
4. Vertical shift by k

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Let's apply these steps to a function!

Complete Transformation Example

Start with $f(x) = x^2$

Transform to: $\boldsymbol{g(x) = -2(x-3)^2 + 5}$

- Shift right 3 units: $(x-3)^2$
- Stretch vertically by 2: $2(x-3)^2$
- Reflect over x-axis: $-2(x-3)^2$
- Shift up 5 units: $-2(x-3)^2 + 5$

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Question: Who can describe how this might look like?

Original Function

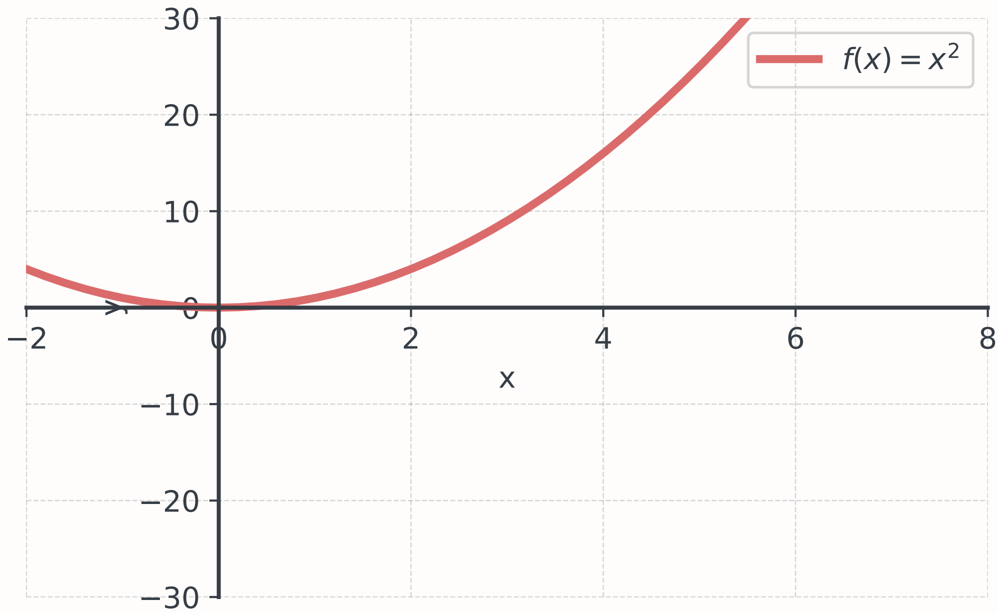

Figure 6: Starting Point: $f(x) = x^2$

Progressive Transformation

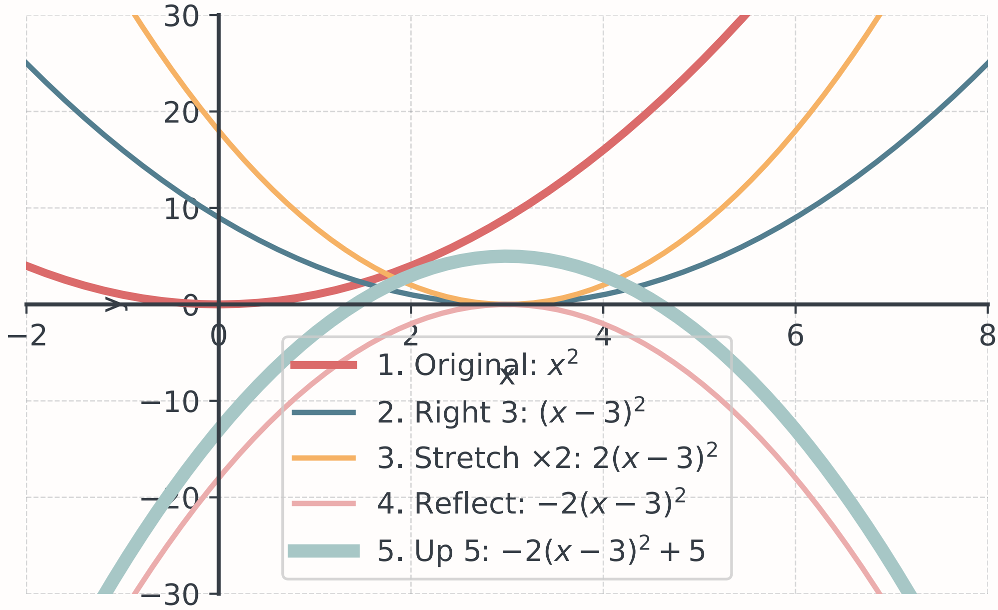

Figure 7: All Transformation Steps

Business Scenario: Market Expansion

Original profit in Germany: $P(x) = -x^2 + 40x - 180$

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Research estimates expansion to France with adjustments:

- 20% higher costs: Multiply by 0.8
- Fixed cost increase of €200: Subtract 200
- 3-month delay: Replace x with $(x - 3)$
- $P_{France}(x) = 0.8[-(x-3)^2 + 40(x-3) - 180] - 200$

...

Question: Should the company expand to France?

Market Expansion Visualization

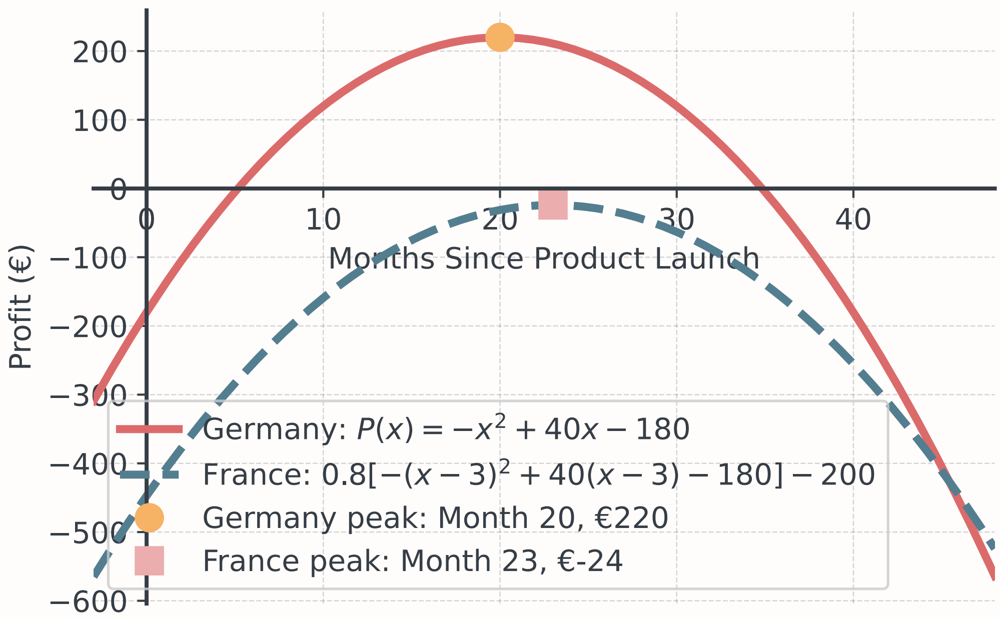

Figure 8: Germany vs France: Profit Comparison

Guided Practice

Individual Exercise Block I

Work alone for 10 minutes, then we discuss the solutions

1. Given $f(x) = x^2 - 4x + 3$, write the equation for:
   a) $f(x)$ shifted up 5 units
   b) $f(x)$ shifted left 2 units
   c) $f(x)$ reflected over the x-axis
2. A cost function is $C(x) = 0.5x^2 + 20x + 500$. Due to inflation:
   - All costs increase by 10%
   - An additional fixed cost of €100 is added
   - Write new cost function and find cost for producing 50 units.

Individual Exercise Block II

Work alone for 5 minutes, then we discuss the solutions

1. The demand for ice cream follows $D(t) = -2(t-7)^2 + 200$ where t is the month.
   a) In which month is demand highest?
   b) If climate change shifts the peak 1 month earlier and increases maximum demand by 15%, write the new function.

Coffee Break - 15 Minutes

Reading Economic Graphs

Extracting Information from Graphs

Graphs tell business stories

Key features to identify:

- Intercepts: Starting values, break-even points
- Slope/Rate of change: Marginal values, trends
- Maximum/Minimum: Optimal points, extremes
- Intersections: Equilibrium, equal values
- Shape: Linear, quadratic, exponential growth patterns
- Domain/Range: Feasible regions, constraints

Graph Analysis Example

Scenario: A graph shows two curves:

- Cost function: Starts at (0, 1000), curves upward
- Revenue function: Starts at origin, curves then flattens
- They intersect at x = 50 and x = 200

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Business insights:

- Fixed costs: €1,000 (y-intercept of cost)
- Break-even points: 50 and 200 units
- Profitable region: Between 50 and 200 units
- Optimal production: Where vertical distance is maximum

Visualizing the Business Story

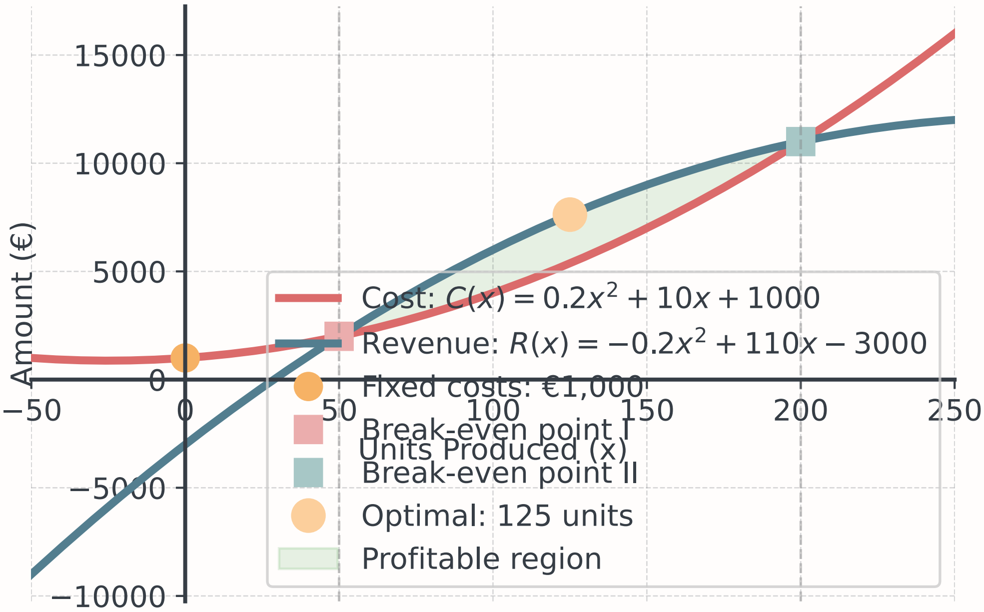

Figure 9: Cost vs Revenue: Finding the Profitable Zone

Comparative Graph Analysis

Comparing multiple scenarios visually

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When comparing functions:

- Parallel lines: Same variable costs, different fixed costs
- Different slopes: Different efficiency levels
- Intersection points: Crossover quantities
- Vertical distances: Profit or loss amounts

Collaborative Problem-Solving

Franchise Expansion Analysis

The Scenario: Coffee Chain Expansion

A successful coffee shop has this profit model for their original location:

$$P(x) = -0.1x^2 + 50x - 4000$$

where x is the number of customers per day.

Understanding the function:

- The $-0.1x^2$ term represents diminishing returns (congestion, slower service)
- The $50x$ term represents revenue per customer
- The -4000 represents daily fixed costs

Now, they want to open franchises in different markets!

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> Transformation Hints
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> - Revenue changes affect the $50x$ coefficient
> - Fixed cost changes affect the -4000 constant
> - Efficiency changes affect the $-0.1x^2$ coefficient (less negative = more efficient)
> - Vertical scaling multiplies the entire function

The Different Markets

A: Tourist Area

- Revenue: 30% higher prices → multiply $50x$ by 1.3
- Fixed costs: Increase by €1,000 → change -4000 to -5000
- Efficiency: Same as original

Task: Apply these transformations to get $P_A(x)$

B: Business District

- Revenue: Same per customer ($50x$ unchanged)
- Efficiency: Better workflow reduces congestion → multiply $-0.1x^2$ by 0.8 (less negative)
- Fixed costs: Increase by €2,000 → change -4000 to -6000

Task: Apply these transformations to get $P_B(x)$

C: University Campus

- Revenue: 40% lower prices → multiply $50x$ by 0.6
- Fixed costs: Decrease by €500 → change -4000 to -3500
- Efficiency: Same as original

Task: Apply these transformations to get $P_C(x)$

Your Tasks

Work in groups of 3-4 students

1. Write the transformed profit function for each location
2. Find the optimal number of customers for each location
3. Calculate the maximum daily profit for each location
4. Which location should be prioritized for expansion and why?
5. If the company can afford total fixed costs of €15,000 per day across all locations (including original), which combination of locations should they operate?

Wrap-Up

Key Takeaways

- Vertical shifts represent fixed changes
- Horizontal shifts represent timing changes
- Stretches and compressions show scaling effects
- Multiple transformations model complex scenarios
- Graphs reveal optimization opportunities
- Business decisions require comprehensive analysis

Final Assessment

5 minutes - Individual work

A retailer's monthly profit is modeled by $P(x) = -2x^2 + 100x - 800$ where x is the number of items sold in hundreds.

Due to economic changes:

- A competitor enters, reducing maximum profit by 25%
- Break-even point shifts from 10 to 15 items (hundreds)

1. What transformation represents the competitor's impact?
2. What transformation represents the break-even shift?
3. Sketch how the graph would change.

Next Session Preview

Session 03-05: Composition, Inverses & Advanced Graphing

- Function composition for multi-step processes
- Finding inverse functions
- When are functions invertible?
- Advanced graphing techniques
- Complex business process modeling

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> Tip
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> Homework Assignment: Complete Tasks 03-04!

Appendix

Comparing All Locations

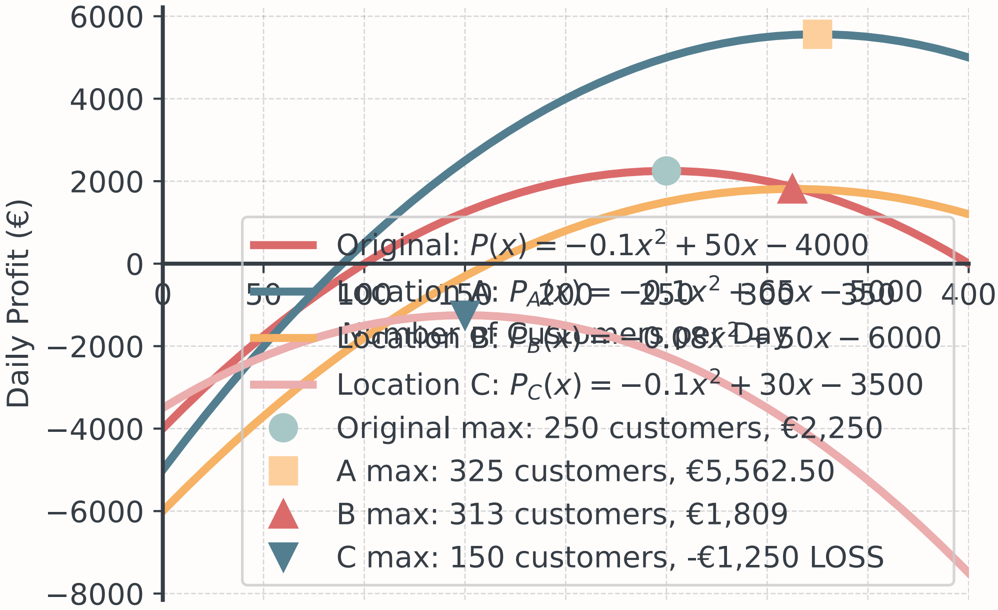

Figure 10: Franchise Profit Comparison: All Four Locations

Location A is the clear winner! Location C never breaks even - avoid it despite lower fixed costs!