Session 03-03 - Quadratic Functions & Basic Optimization

Section 03: Functions as Business Models

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Entry Quiz - 10 Minutes

Review from Session 03-02

Work individually, then we discuss together as group

1. Find the market equilibrium for:
   - Demand: $Q_d = 200 - 2p$
   - Supply: $Q_s = 50 + 3p$
2. Write the equation of a line passing through points (2, 8) and (5, 20).
3. For the cost function $C(x) = 500 + 12x$ and revenue $R(x) = 25x$, find the profit when $x = 100$.

Homework Review - 20 Minutes

Discussing Tasks 03-02

Let's discuss the most difficult tasks from last lecture

- Problem 5: Market competition analysis
  - How did you determine the break-even data usage?
- Problem 6: Production planning with constraints
  - Challenges with multiple constraints?
- Problem 7: Dynamic pricing (if attempted)
  - What price seemed optimal in your testing?

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> **Note**
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> Today we'll learn the exact method to find that optimal price!

Introduction to Quadratic Functions

From Linear to Quadratic

Quadratic functions model accelerating change

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Linear vs. Quadratic:

- Linear: $f(x) = mx + b \longrightarrow$ Constant rate of change
- Quadratic: $f(x) = ax^2 + bx + c \longrightarrow$ Changing rate of change
- Graph shape: Quadratic $\longrightarrow$ Parabola (U-shaped or ∩-shaped)
- Business meaning:
  - Linear $\longrightarrow$ Fixed relationships
  - Quadratic $\longrightarrow$ Optimization opportunities!

Standard Form

The foundation: f(x) = ax^2 + bx + c

Key components:

- a: Direction and width
  - $a > 0$: Opens upward (has minimum)
  - $a < 0$: Opens downward (has maximum)
  - $|a|$ larger $\longrightarrow$ Narrower parabola
- b: Affects position of vertex
- c: y-intercept (value when $x = 0$)

Example: Profit Function

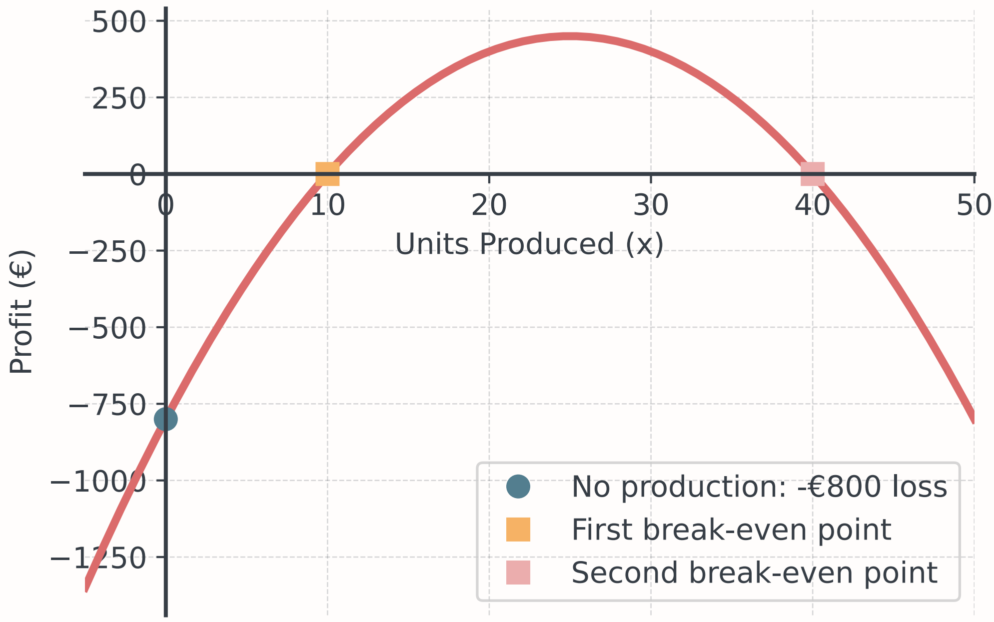

Figure 1: Quadratic Profit Function: $P(x) = -2x^2 + 100x - 800$

Quick Practice - 10 Minutes

Work individually, then we discuss

- Determine: Does it open upward (U) or downward (∩)?
- Determine: Does it have a maximum or minimum?
- Determine: What is the y-intercept?

a) $R(x) = -3x^2 + 120x - 500$

b) $C(x) = 2x^2 + 40x + 1000$

c) $P(x) = -x^2 + 50x - 300$

Challenge: For c. find the break-even points.

Break - 10 Minutes

Finding the Vertex

The Vertex Formula

The key: x = -b/2a

For $f(x) = ax^2 + bx + c$:

- Vertex x-coordinate: $x_v = -\frac{b}{2a}$
- Vertex y-coordinate: $f(x_v) = f(-\frac{b}{2a})$
- Vertex represents:
  - Maximum if $a < 0$ (parabola opens down)
  - Minimum if $a > 0$ (parabola opens up)
- Axis of symmetry: Vertical line $x = x_v$

Vertex Example: Revenue Optimization

A company's revenue depends on price:

$$R(p) = -50p^2 + 2000p$$

- Find optimal price: $p_v = -\frac{2000}{2(-50)} = -\frac{2000}{-100} = 20$ euros
- Maximum revenue: $R(20) = 20000$ euros
- Interpretation: Charging €20 maximizes revenue at €20,000

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> **Tip**
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> The axis of symmetry divides the parabola into mirror images. Points equidistant from it have equal revenue!

Visualization

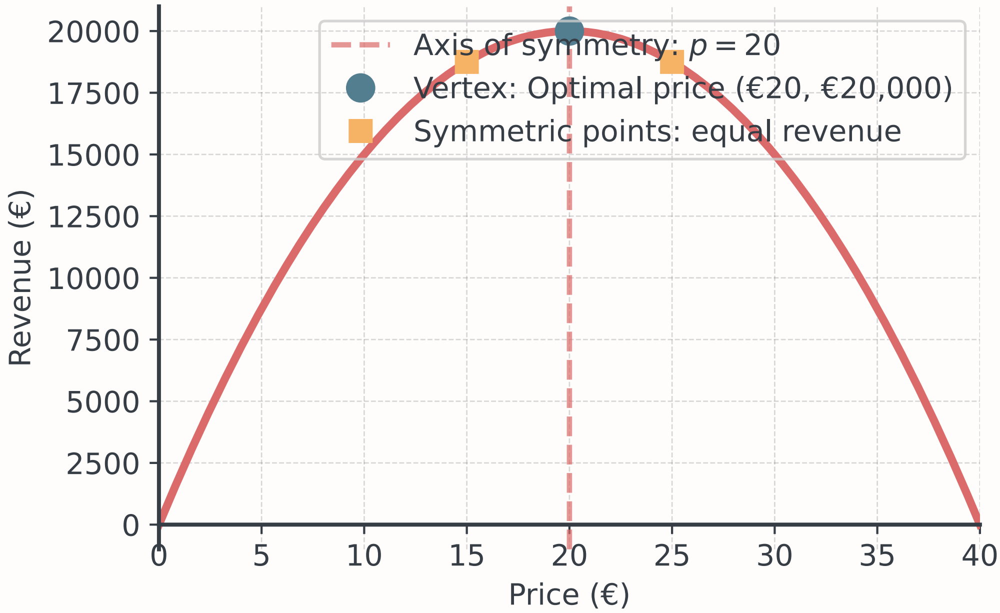

Figure 2: $R(p) = -50p^2 + 2000p$ with Vertex and Axis of Symmetry

Vertex Form

Alternative representation: f(x) = a(x - h)^2 + k

- Vertex: (h, k) - directly visible!
- Direction: a (same as standard form)
- Advantage: Vertex immediately apparent
- Transformation from vertex:
  - Horizontal shift by h
  - Vertical shift by k
- Example: $f(x) = 2(x - 3)^2 + 5 \longrightarrow$ Vertex at $(3, 5)$, minimum
- Example: $g(x) = -(x + 4)^2 + 10 \longrightarrow$ Vertex at $(-4, 10)$, maximum

Completing the Square

Converting to Vertex Form

Transform $f(x) = ax^2 + bx + c$ to $f(x) = a(x - h)^2 + k$

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Process:

1. Factor out a from first two terms
2. Complete the square inside parentheses
3. Simplify to vertex form

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ℹ Note

Sorry, I know I said we don't need that!

Step-by-Step Example

Convert $\boldsymbol{f(x) = 2x^2 - 12x + 10}$ to vertex form

1. Factor out 2: $f(x) = 2(x^2 - 6x) + 10$
2. Complete square: Need $\left(\frac{-6}{2}\right)^2 = 9$
3. Add and subtract: $f(x) = 2(x^2 - 6x + 9 - 9) + 10$
4. Rewrite: $f(x) = 2((x - 3)^2 - 9) + 10$
5. Distribute: $f(x) = 2(x - 3)^2 - 18 + 10$
6. Final form: $f(x) = 2(x - 3)^2 - 8$
7. Vertex: $(3, -8)$ with minimum value −8

Fast Exercise

Solve in 5 minutes, then we compare solutions

Convert $f(x) = 3x^2 + 18x + 20$ to vertex form by completing the square.

Business Applications

Price-Dependent Demand

When price affects quantity: Revenue becomes quadratic!

Basic Scenario:

- Demand function: $Q = a - bp$ (quantity depends on price)
- Revenue: $R = p \times Q = p(a - bp)$
- Expanded: $R(p) = ap - bp^2 = -bp^2 + ap$
- This is quadratic in p!

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Tip

Remember, we have seen this in the past!

Example: Concert Venue

A venue (capacity: 1000) has ticket demand: $Q = 1000 - 20p$

- Revenue function: $R(p) = p(1000 - 20p) = 1000p - 20p^2$
- Optimal price: $p^* = -\frac{1000}{2(-20)} = \frac{1000}{40} = 25$ euros
- Tickets sold: $Q = 1000 - 20(25) = 500$
- Maximum revenue: $R(25) = 25 \times 500 = 12,500$

- At €0: Demand = 1000 (full capacity if free)
- At €50: Demand = 0 (too expensive, no one buys)

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> ⚠ Warning
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> Note: This maximizes revenue, not necessarily profit!

Guided Practice - 20 Minutes

Individual Exercise Block

Work alone for 15 minutes, then we compare solutions

1. For $f(x) = x^2 - 8x + 12$:
   a) Find the vertex using the formula
   b) Determine if it's a maximum or minimum and find the y-intercept
2. A profit function is $P(x) = -3x^2 + 240x - 3600$:
   a) Find the number of units that maximizes profit
   b) Calculate the maximum profit and the break-even points
3. Convert $f(x) = 2x^2 - 12x + 14$ to vertex form by completing the square, then identify the vertex.

Coffee Break - 15 Minutes

Projectile Motion

Product Launch Campaign

Marketing models new product awareness like projectile motion

$$A(t) = -2t^2 + 24t$$

where A is awareness score and t is weeks after launch.

- Peak awareness time: $t = -\frac{24}{2(-2)} = 6$ weeks
- Maximum awareness: $A(6) = -72 + 144 = 72$ points
- Campaign ends when $A(t) = 0$: at $t = 0$ and $t = 12$ weeks

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> Tip
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> Campaign follows symmetric pattern: builds to peak at 6 weeks, then decays at same rate.

Campaign Awareness

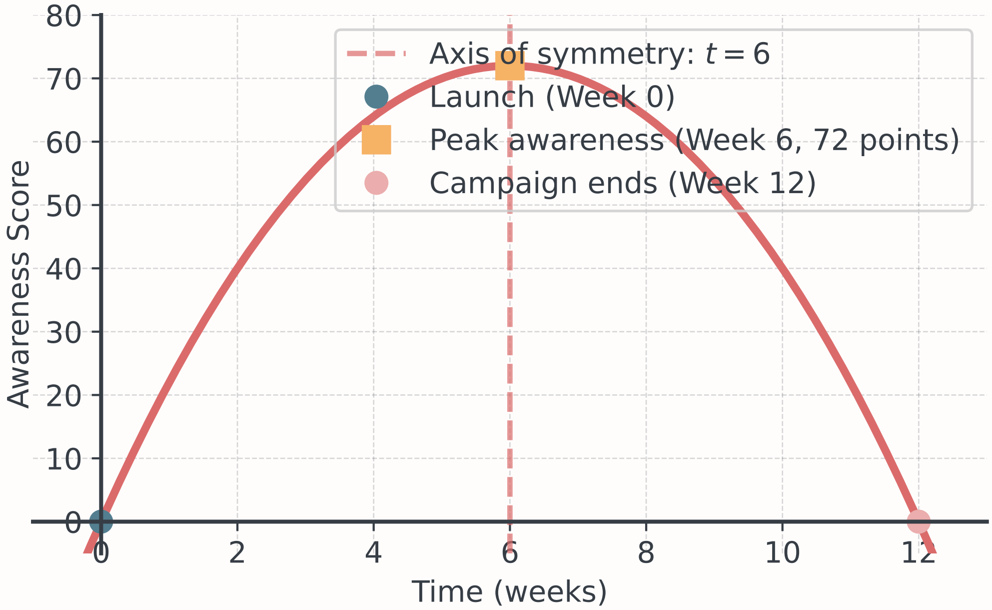

Figure 3: Product Launch Campaign: $A(t) = -2t^2 + 24t$

Area Optimization

Maximizing Area with Constraints

Classic problem: Maximum area with fixed perimeter

Rectangular Storage Area with 200 meters of fencing available. One side against a building (no fence) and we want to maximize storage area.

- Let x = width, y = length parallel to building
- Constraint: $2x + y = 200$ (fencing)
- So: $y = 200 - 2x$
- Area: $A = xy = x(200 - 2x) = 200x - 2x^2$
- Maximum at: $x = -\frac{200}{2(-2)} = 50$ meters
- Dimensions: 50m × 100m, Area = 5000 m^2

Visualization

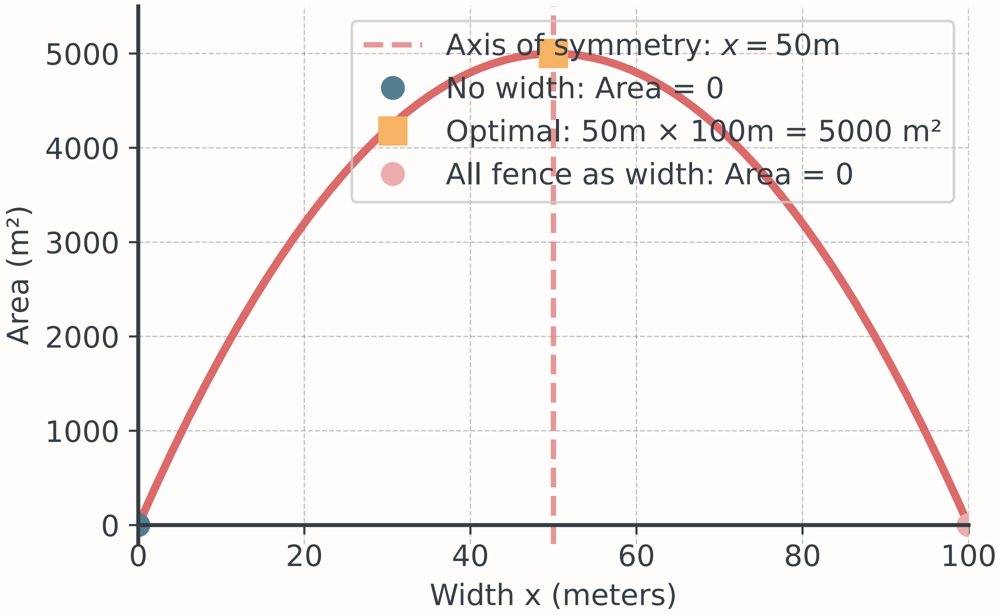

Figure 4: Maximizing Area: $A(x) = 200x - 2x^2$

Symmetric design: Too narrow OR too wide both reduce area - optimal is exactly in the middle!

Collaborative Problem-Solving - 30 Minutes

Comprehensive Business Optimization

The Scenario: Smart Tech Product Launch

Smart Tech is launching a new tablet. Market research indicates:

- At €200: would sell 8,000 units per month
- At €400: would sell 4,000 units per month
- At €600: would sell 0 units (too expensive)
- Production cost: €150 per tablet
- Fixed monthly costs: €200,000

Assume linear demand relationship.

Your Tasks:

Work in groups of 3-4 students

1. Derive the demand function $Q(p)$ where p is price
2. Express revenue $R(p)$ as a function of price (this will be quadratic!)
3. Find the price that maximizes revenue
4. Express profit $\Pi(p)$ as a function of price

5. Find the price that maximizes profit (different from revenue-maximizing price!)
6. If the company can only produce 5,000 tablets per month, should they use the profit-maximizing price? Explain.

Wrap-Up

Key Takeaways

- The vertex formula $x = -\frac{b}{2a}$ is your optimization tool
- Quadratic functions model scenarios with changing rates
- Maximum/minimum depends on sign of a
- Revenue maximization ≠ Profit maximization
- Completing the square reveals the vertex form
- Real constraints may override mathematical optima

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> Remember
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> Every parabola has a minimum or a maximum point!

Final Assessment

5 minutes - Individual work

A small bakery's daily profit for chocolate cakes is modeled by:

$$P(x) = -x^2 + 14x - 33$$

where x is the price in euros.

1. Find the price that maximizes profit
2. Calculate the maximum daily profit
3. Find the break-even prices

Next Session Preview

Session 03-04: Transformations & Graphical Analysis

- Shifting functions horizontally and vertically
- Stretching and reflecting graphs
- Reading graphs to understand business scenarios
- Function composition in business contexts
- Multiple representation mastery

> Tip
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> Homework Assignment: Complete Tasks 03-03!