Session 02-05 - Exponential, Logarithmic & Complex Word Problems

Section 02: Equations & Problem-Solving Strategies

Dr. Nikolai Heinrichs & Dr. Tobias Vlćek

Entry Quiz

Test Your Equation-Solving Skills

10 minutes - individual work, then peer review

a) Solve: $\frac{x+2}{x^2-4} + \frac{1}{x-2} = \frac{3}{x+2}$

b) Solve: $\sqrt{x+5} - \sqrt{x-3} = 2$

c) Solve: $x^3 - 3x^2 - x + 3 = 0$ (hint: try grouping)

d) If $2^{x+1} = 3 \cdot 2^x - 4$, find x

e) Express as a single logarithm: $2\log_3(x) - \log_3(x+2) + \log_3(3)$

. . .

> Tip
>
> These combine all our equation types - prepare for today's problems!

Homework Presentations

Solutions from Tasks 02-04

20 minutes - presentation and discussion

- Present difficult problems
- Discuss investment return calculations
- Share strategies for work rate problems
- Review any challenging radical equations

. . .

> Note
>
> Today we solve the most complex equation types - perfect preparation for your assessment!

Key Focus Today

Beyond Basic Properties

We already know logarithm properties from Session 01-05

Today's NEW focus:

- Solving exponential equations with different bases
- Solving logarithmic equations with multiple logs
- Converting between forms strategically
- Handling equations that mix types
- Tackling complex multi-step word problems

. . .

> ! Important
>
> Today is all about application and problem-solving!

Break - 10 Minutes

Advanced Exponential Equations

Equations with Different Bases

When you can't make bases equal

Solve: $3^x \cdot 5^{x-1} = 45$

- Rewrite: $3^x \cdot \frac{5^x}{5} = 45$
- Simplify: $\frac{3^x \cdot 5^x}{5} = 45$
- Combine: $\frac{(3 \cdot 5)^x}{5} = 45$
- So: $15^x = 225 = 15^2$
- Therefore: $x = 2$

. . .

> Tip
>
> Look for ways to combine or separate bases strategically!

Mixed Exponential Systems

Solving simultaneous exponential equations

$2^x + 2^y = 12$

$2^x - 2^y = 4$

- Let $u = 2^x$ and $v = 2^y$ for simplicity

- System becomes: $u + v = 12$ and $u - v = 4$
- Add equations: $2u = 16$, so $u = 8$
- Subtract: $2v = 8$, so $v = 4$
- Therefore: $2^x = 8 = 2^3$, giving $x = 3$
- And: $2^y = 4 = 2^2$, giving $y = 2$

Exponential Inequalities

New territory: solving inequalities

Solve: $2^{x+1} > 8^{x-1}$

- Rewrite right side: $8^{x-1} = \left(2^3\right)^{x-1} = 2^{3(x-1)}$
- Inequality becomes: $2^{x+1} > 2^{3x-3}$
- Since base 2 > 1, we can compare exponents directly
- $x + 1 > 3x - 3$
- $4 > 2x$
- Solution: $x < 2$

Advanced Logarithmic Equations

Equations with Mixed Bases

Using change of base strategically

Solve: $\log_2(x) \cdot \log_x(8) = 3$

- Use change of base: $\log_x(8) = \frac{\log_2(8)}{\log_2(x)} = \frac{3}{\log_2(x)}$
- Substitute: $\log_2(x) \cdot \frac{3}{\log_2(x)} = 3$
- Simplify: $3 = 3$ ✓
- This is always true for any valid x!
- Domain restriction: $x > 0, x \neq 1$
- Solution: $x \in (0, 1) \cup (1, \infty)$

Logarithmic Systems

Multiple equations with logs

$\log(x) + \log(y) = 2$

$\log(x) - \log(y) = 1$

- Add equations: $2\log(x) = 3$, so $\log(x) = 1.5$
- Therefore: $x = 10^{1.5} = 10\sqrt{10}$
- Subtract second from first: $2\log(y) = 1$
- So: $\log(y) = 0.5$, thus $y = \sqrt{10}$
- Check: $\log\left(10\sqrt{10}\right) + \log\left(\sqrt{10}\right) = 1.5 + 0.5 = 2$ ✓

Complex Word Problems

Investment Comparison

Clean compound interest problem

Two investments of €5,000 each. Investment A earns 8% annually. Investment B earns rate r annually. After 3 years, their combined value is €12,000. Find rate r for Investment B.

- Investment A after 3 years: $5000(1.08)^3$ = 6298.56$
- Combined value equation: $6298.56 + 5000(1 + r)^3 = 12000$
- Simplify: $5000(1 + r)^3 = 5701.44$ and then $(1 + r)^3 = 1.140288$
- Take cube root: $1 + r = \sqrt[3]{1.140288}$
- Using logs: $1 + r = e^{\frac{\ln(1.140288)}{3}} = e^{0.0438} \approx 1.0448$
- Therefore: $r = 0.0448$ or about 4.5%

Population Competition Model

Ecological application with constraints

Two species compete for resources. Species A grows exponentially at 10% per year. Species B starts with twice the population but grows at rate $r\%$. After 10 years, they have equal populations. Find r and the population ratio after 5 years.

- After 10 years: $P(1.1)^{10} = 2P(1 + r)^{10}$
- Simplify: $(1.1)^{10} = 2(1 + r)^{10}$
- $(1 + r)^{10} = \frac{(1.1)^{10}}{2}$
- $1 + r = \left(\frac{1.1^{10}}{2}\right)^{1/10} = \frac{1.1}{2^{0.1}}$
- $r \approx 0.033$ or 3.3%, after 5 years $

Guided Practice

Mixed Equation Types

Work independently

1. Solve: $3^{2x} - 4 \cdot 3^x + 3 = 0$
2. $\{2^x \cdot 3^y = 72$

   $x + y = 5$
3. A bacteria culture grows at 20% per hour. Under treatment, the hourly growth factor is reduced by a constant proportion r (so each hour the factor is $1.2(1 - r)$). After 5 hours, the treated population is 80% of the untreated population. Find r.

Coffee Break - 15 Minutes

Multi-Step Problem Solving

The IDEA Method in Action

Identify - Develop - Execute - Assess

A company's revenue follows $R = 100(1 - e^{-kt})$ million €, where t is years since launch. After 2 years, revenue is €40 million. Find k.

- $40 = 100(1 - e^{-2k})$
- $0.4 = 1 - e^{-2k}$
- $e^{-2k} = 0.6$
- $k = -\frac{\ln(0.6)}{2} \approx 0.255$
- Check: $R(2) = 100(1 - e^{-0.51}) \approx 40$ ✓

Collaborative Problem-Solving

Epidemic Modeling Challenge

Work in groups

A disease spreads through a population of 10,000. The infected count follows:

$$I = \frac{10000}{1 + 99e^{-0.5t}}$$

a) How many are initially infected?
b) When will half the population be infected?
c) If a vaccine reduces the spread rate by 40%, modify the model

...

> Tip
>
> This is a logistic growth model - different from pure exponential!

Wrap-up

Key Takeaways

Essential skills for the mock exam

- Exponential solving: Use logs strategically, not just mechanically
- Logarithmic solving: Check domains and validate solutions
- Mixed equations: Recognize when to use substitution
- Systems: Sometimes have no solution or infinitely many
- Word problems: Build models systematically
- Complex scenarios: Break into manageable steps

- Verification: Always check solutions make practical sense

Final Assessment

10 minutes - individual assessment

1. Solve: $4^x - 3 \cdot 2^x = -2$
2. Solve: $\log_2(x+3) - \log_4(x) = 2$
3. An investment doubles in 8 years at rate r%, then the rate increases to $(r+2)$%. How long for it to triple from the original amount?

Next Session Preview

Session 02-06: Foundation Assessment

MINI-MOCK EXAM 1

- 90 minutes under exam conditions
- Covers everything from Section 01 and 02
- Mix of pure math and applications
- About 8-10 problems of varying difficulty

. . .

> **! Important**
>
> Review all equation-solving methods!