Session 02-04 - Fractional, Radical & Cubic Equations

Section 02: Equations & Problem-Solving Strategies

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Entry Quiz

Quick Review of Previous Methods

10 minutes - individual work, then peer review

a) Solve by factoring: $x^2 - 9x + 20 = 0$

b) Use the quadratic formula: $2x^2 + 3x - 1 = 0$

c) Solve the biquadratic: $x^4 - 5x^2 + 4 = 0$

d) Find the discriminant of: $x^2 - 6x + 9 = 0$

e) Factor completely: $x^3 - 8$

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> **Tip**
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> Check your factoring skills - they're crucial for today!

Homework Presentations

Solutions from Tasks 02-03

20 minutes - presentation and discussion

- Share your break-even analysis solutions
- Discuss method selection strategies
- Present any challenging biquadratic equations
- Review work rate problems if time permits

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> **Note**
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> Today we extend our toolkit to handle more complex equation types!

Key Concept Review

Building on What You Know

Connecting to today's content

Your current equation-solving toolkit:

- Zero Product Property: If $AB = 0$, then $A = 0$ or $B = 0$
- Quadratic methods: Factoring, formula, completing square
- Substitution: Transform complex equations to simpler ones

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Today's additions:

- Domain restrictions for rational equations
- Checking for extraneous solutions
- Advanced factoring for cubics

Fractional Equations

Understanding Domain Restrictions

Critical new concept

Domain: The set of all valid input values

For fractional equations, we must exclude values that make denominators zero!

Example: $\frac{1}{x-3}$ has domain restriction: $x \neq 3$

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> ⚠ Always Check First!
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> Before solving any fractional equation, identify ALL domain restrictions.

Solving Fractional Equations

Method: Clear denominators by multiplying by LCD

Example: Solve $\frac{2}{x} + \frac{3}{x-1} = 1$

1. Find domain restrictions: $x \neq 0, x \neq 1$
2. Find LCD: $x(x-1)$
3. Multiply all terms by LCD: $2(x-1) + 3x = x(x-1)$
4. Expand: $2x - 2 + 3x = x^2 - x$
5. Rearrange: $x^2 - 6x + 2 = 0$
6. Solve: $x = \frac{6 \pm \sqrt{36-8}}{2} = 3 \pm \sqrt{7}$
7. Check domain: Both solutions valid! ✓

Common Fractional Patterns

Recognize these structures

Simple Fractions

$\frac{a}{x} + \frac{b}{x} = c$

- Combine: $\frac{a+b}{x} = c$
- Solve: $x = \frac{a+b}{c}$

Cross Multiplication

$\frac{a}{b} = \frac{c}{d}$

- Cross multiply: $ad = bc$
- Much faster than finding LCD!

Complex Fractions

$\frac{\frac{a}{x}}{\frac{b}{y}} = c$

- Simplify: $\frac{ay}{bx} = c$
- Then solve normally

Application: Work Rate Problem I

Practical example with domain restrictions

Two teams can complete a project. Their combined work rate equation is:

$$\frac{1}{x} + \frac{1}{x+5} = \frac{1}{3}$$

where x is the time (in days) for Team A to complete the project alone.

- Domain restrictions: $x > 0$ (time must be positive)
- Also $x \neq -5$, but this is already excluded by $x > 0$
- Solving: Find LCD = $3x(x+5)$

Application: Work Rate Problem II

- Multiply through: $3(x+5) + 3x = x(x+5)$
- Expand: $3x + 15 + 3x = x^2 + 5x$
- Rearrange: $x^2 - x - 15 = 0$
- Using quadratic formula: $x = \frac{1 \pm \sqrt{1+60}}{2} = \frac{1 \pm \sqrt{61}}{2}$
- Solutions: $x \approx 4.41$ or $x \approx -3.41$
- Check domain: Only $x = 4.41$ days is valid (positive time)
- Business meaning: Team A takes 4.41 days alone, Team B takes 9.41 days alone
- Together they complete it in 3 days as required

Break - 10 Minutes

Radical Equations

Solving Strategy

Isolate, square, check!

Key principle: To eliminate a square root, square both sides

Critical warning: Squaring can introduce extraneous solutions!

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Example: $\sqrt{x+3} = x - 1$

- Square both sides: $x + 3 = (x-1)^2$
- Expand: $x + 3 = x^2 - 2x + 1$
- Rearrange: $x^2 - 3x - 2 = 0$
- Solutions: $x = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$
- $x_1 = \frac{3+\sqrt{17}}{2} \approx 3.56$, $x_2 = \frac{3-\sqrt{17}}{2} \approx -0.56$

Checking for Extraneous Solutions

Essential verification step

Check $\boldsymbol{x_1 \approx 3.56}$:

- Left: $\sqrt{3.56 + 3} = \sqrt{6.56} \approx 2.56$
- Right: $3.56 - 1 = 2.56$ ✓

Check $\boldsymbol{x_2 \approx -0.56}$:

- Left: $\sqrt{-0.56 + 3} = \sqrt{2.44} \approx 1.56$
- Right: $-0.56 - 1 = -1.56$ ✗

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⚠ Warning

Only x_1 is valid! Always check radical equation solutions!

Multiple Radicals

More complex scenarios

Solve: $\sqrt{x+5} + \sqrt{x} = 5$

- Isolate one radical: $\sqrt{x+5} = 5 - \sqrt{x}$
- Square: $x + 5 = 25 - 10\sqrt{x} + x$
- Simplify: $5 = 25 - 10\sqrt{x}$
- Isolate: $10\sqrt{x} = 20$, so $\sqrt{x} = 2$
- Solution: $x = 4$

- Check: $\sqrt{9} + \sqrt{4} = 3 + 2 = 5$ ✓

Cubic Equations

Factoring Strategies

Building on what you know

Special Forms

You already know these from Section 01:

- Sum of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- Difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example: $x^3 - 8 = x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)$

Trial and Error

For simple cubics, try small integer values:

- Test $x = 0, \pm 1, \pm 2, \pm 3 \ldots$
- If $x = a$ works, then $(x - a)$ is a factor
- Use long division to find the other factor

Grouping

For $x^3 + px^2 + qx + r$:

- Look for common factors first
- Try grouping in pairs
- Example: $x^3 + 2x^2 - x - 2 = x^2(x + 2) - 1(x + 2) = (x + 2)(x^2 - 1)$

Solving Cubic Equations

Step-by-step approach

Solve: $x^3 - 6x^2 + 11x - 6 = 0$

- Try simple values: Test $x = 1$: $1 - 6 + 11 - 6 = 0$ ✓
- So $(x - 1)$ is a factor! Now we need to find what it multiplies with
- Divide: We know $x^3 - 6x^2 + 11x - 6 = (x - 1) \times ?$
- By inspection or long division: $(x - 1)(x^2 - 5x + 6) = 0$
- Factor the quadratic: $(x - 1)(x - 2)(x - 3) = 0$
- Solutions: $x = 1, 2, 3$

Special Case: Difference/Sum of Cubes

Using what you already know

Solve $x^3 - 27 = 0$

- Recognize as difference of cubes: $x^3 - 3^3 = 0$
- Factor: $(x - 3)(x^2 + 3x + 9) = 0$

- From first factor: $x = 3$
- From second factor: Use quadratic formula
- $x = \frac{-3 \pm \sqrt{9-36}}{2} = \frac{-3 \pm \sqrt{-27}}{2}$ (no real solutions)
- Only real solution: $x = 3$

Guided Practice

Individual Exercises

Work independently

1. Solve and state domain: $\frac{3}{x-2} + \frac{1}{x} = 1$
2. Solve: $\sqrt{2x+1} = x - 2$
3. Factor and solve: $x^3 - 27 = 0$
4. Solve: $\frac{x}{x+1} = \frac{2}{x-1}$
5. Solve: $\sqrt{x+7} - \sqrt{x} = 1$
6. Solve: $x^3 + 2x^2 - 5x - 6 = 0$

Coffee Break - 15 Minutes

Application & Extension

Production Rate Problem

Find the individual times.

Machine A can complete an order in x hours. Machine B takes 3 hours longer. Working together, they complete it in 2 hours.

- A's rate: $\frac{1}{x}$ orders/hour, B's rate: $\frac{1}{x+3}$ orders/hour
- Combined: $\frac{1}{x} + \frac{1}{x+3} = \frac{1}{2}$
- Solve: $2(x+3) + 2x = x(x+3)$
- $x^2 - x - 6 = 0$
- $(x-3)(x+2) = 0$
- Since $x > 0$: Machine A takes 3 hours, B takes 6 hours

Investment Growth

Compound interest with radicals

An investment grows according to: $A = P\sqrt{1+0.2t}$

If €1,000 grows to €1,500, find the time t.

- Set up: $1500 = 1000\sqrt{1+0.2t}$
- Simplify: $\sqrt{1+0.2t} = 1.5$
- Square: $1 + 0.2t = 2.25$
- Solve: $0.2t = 1.25$

- Time: $t = 6.25$ years

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⚠ Warning

Do we have to check the domain restrictions?

Collaborative Problem-Solving

Complex Rate Challenge

Work in pairs to solve this problem

A chemical reaction follows the rate equation:

$$\frac{C}{t} + \frac{C}{t+2} = 3$$

where C is concentration and t is time in hours.

a) Find the time when this relationship holds for $C = 6$
b) Verify your solution makes physical sense

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Tip

Consider domain restrictions and check all solutions!

Spot the Error

Find and fix the mistake

A student solved $\sqrt{x+4} = x - 2$:

> Square both sides with $x + 4 = x - 2$, therefore: $4 = -2$, which is impossible. So there's no solution.

What went wrong?

Wrap-up

Key Takeaways

Master these essential concepts

- Domain restrictions must be checked FIRST in rational equations
- LCD method clears fractions efficiently
- Squaring introduces extraneous solutions - always verify!

- Cubic factoring uses rational root theorem and special forms
- Cross multiplication works for proportion equations
- Real-world rates often involve rational equations

Final Assignment

10 minutes - individual assessment

Solve the following:

a) $\frac{2}{x+1} - \frac{1}{x-1} = \frac{1}{2}$

b) $\sqrt{3x-2} = x$

c) Factor: $x^3 + 8$

Next Session Preview

Session 02-05: Exponential, Logarithmic & Complex Word Problems

- Exponential growth and decay models
- Logarithmic equations and applications
- Complex multi-step word problems
- Financial mathematics applications

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> ! Important
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> Review logarithm properties from Section 01 - we'll use them extensively!

Homework Assignment

Complete Tasks 02-04

Focus on:

- Domain restriction practice
- Checking all solutions in radical equations
- Factoring cubic expressions
- Real-world rate problems

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> Tip
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> Remember: In fractional equations, always identify restrictions before solving!