Session 02-03 - Quadratic & Biquadratic Equations

Section 02: Equations & Problem-Solving Strategies

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Entry Quiz

Quick Review of Essential Skills

10 minutes - individual work, then peer review

a) Factor completely: $x^2 - 7x + 12$

b) Factor by grouping: $2x^3 - 6x^2 + x - 3$

c) Solve the system: $\{2x + y = 10$ $x - y = 2$

d) Complete the square: $x^2 + 6x + ?$

e) Identify a, b, c in: $3x^2 - 2x + 5 = 0$

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> Tip
>
> These skills are essential for today's methods!

Homework Presentations

Solutions from Tasks 02-02

30 minutes - presentation and discussion

- Present your most challenging problem
- Share alternative solution methods
- Discuss any conceptual difficulties
- Ask questions about problems you struggled with

Key Concepts

Equation Types Overview

Today's new topics:

- Linear: $ax + b = 0 \longrightarrow$ One solution

- Quadratic: $ax^2 + bx + c = 0 \longrightarrow$ Up to two solutions
- Biquadratic: $ax^4 + bx^2 + c = 0 \longrightarrow$ Up to four solutions

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> **ℹ Why equal to zero?**
>
> Good question! Either we want to determine the intersection of the graph and the x-axis (hence y=0) or we try to make an equation equal to zero to determine the value of x easily.

Solving Equations

Zero Form & Linear Equations

The Zero Product Property

If $A \cdot B = 0$, then $A = 0$ or $B = 0$

Example: Solve $3x - 6 = 0$

- Factor: $3(x-2) = 0$
- Apply property: $x - 2 = 0$
- Solution: $x = 2$

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> **ℹ Note**
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> This principle extends to all equation types!

Three Methods for Quadratics

Let's solve the same equation three ways: $x^2 - 5x + 6 = 0$

Factoring

When to use: Integer coefficients, factorable, fastest

- Factor: $(x-2)(x-3) = 0$
- Apply Zero Product Property
- Solutions: $x = 2$ or $x = 3$

Quadratic Formula

When to use: Always works, but is slower

- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $a = 1$, $b = -5$, $c = 6$
- $x = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2}$
- Solutions: $x = 3$ or $x = 2$

Completing Square

When to use: Only in special cases (my recomendation)

- $x^2 - 5x = -6$
- $x^2 - 5x + \frac{25}{4} = -6 + \frac{25}{4} = \frac{1}{4}$
- $\left(x - \frac{5}{2}\right)^2 = \frac{1}{4}$
- $x - \frac{5}{2} = \pm\frac{1}{2}$
- Solutions: $x = 3$ or $x = 2$

The Discriminant

For $ax^2 + bx + c = 0$, the discriminant $\Delta = b^2 - 4ac$ tells us:

Δ Value	Solution Type	Graph Behavior	Factorability
$\Delta > 0$ and perfect square	Two rational solutions	Crosses x-axis twice	Easily factorable
$\Delta > 0$ but not perfect square	Two real (irrational) solutions	Crosses x-axis twice	Not factorable over integers
$\Delta = 0$	One repeated real solution	Touches x-axis once	Perfect square factorization
$\Delta < 0$	No real solutions	Doesn't touch x-axis	Not factorable over reals

Method Selection Guide

Which method should you use?

Quadratic Equation: $ax^2 + bx + c = 0$

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Calculate Δ = b² - 4ac
│
├─ Δ < 0 → No real solutions
│
├─ Δ = 0 → One solution: x = -b/(2a) (Perfect square trinomial)
│
└─ Δ > 0 → Two real solutions
           │
           └─ Is Δ a perfect square?
              │
              ├─ YES → Try factoring first
              │
              └─ NO → Use quadratic formula
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> ℹ Note
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> Interested in more details and the origin of the quadratic formula? Head over here

Biquadratic Equations

Extending to fourth-degree

Form: $ax^4 + bx^2 + c = 0$

Strategy: Substitution!

- Let $u = x^2$
- Solve $au^2 + bu + c = 0$
- Back-substitute to find x

Solving Biquadratic Equations

Let's Look at an Example

Example: $x^4 - 5x^2 + 4 = 0$

- Let $u = x^2$: $u^2 - 5u + 4 = 0$
- Factor: $(u - 1)(u - 4) = 0$
- So $u = 1$ or $u = 4$
- If $x^2 = 1$: $x = \pm 1$
- If $x^2 = 4$: $x = \pm 2$
- Four solutions: $x = -2, -1, 1, 2$

Guided Practice

Individual Exercises

Work independently, then we'll discuss

1. Solve: $(2x - 6)(x + 4) = 0$
2. Solve: $5x - 15 = 0$
3. Solve by factoring: $x^2 + 7x + 10 = 0$
4. Use quadratic formula: $2x^2 - 3x - 2 = 0$
5. Complete the square: $x^2 - 4x - 5 = 0$
6. Solve: $x^4 - 13x^2 + 36 = 0$

Break - 10 Minutes

Practice Session

Practice Set A

10 minutes - Fundamentals

1. Solve for x:

a) $4x - 12 = 0$
b) $-3x + 15 = 0$
c) $\frac{2x-8}{4} = 3$

2. Without solving, determine the number of real solutions:

a) $x^2 + 4x + 4 = 0$
b) $x^2 - 3x + 5 = 0$
c) $3x^2 - 12x + 9 = 0$

Practice Set B: Core Skills

5 minutes - Individually

Solve each using the most efficient method and justify your choice:

a) $x^2 - 11x + 30 = 0$
b) $2x^2 + 5x - 3 = 0$
c) $x^2 - 6x + 9 = 0$
d) $3x^2 - 7x + 1 = 0$

Practice Set C: Challenge Problems

1. Solve these equations:

a) $(x^2 - 4)(x^2 - 9) = 0$
b) $x^3 - 4x^2 - 5x + 20 = 0$
c) $(x - 1)^2(x + 3) = 0$

2. A factory produces custom widgets. The cost function is $C = x^2 - 40x + 500$ and the revenue function is $R = 20x$, where x is the number of units.

a) Find the profit function $P = R - C$
b) At what production levels does the factory break even?

Practice Set D: Patterns

5 minutes - Individually

Solve these related equations and find the pattern:

a) $x^2 - 5x + 6 = 0$
b) $x^2 - 5x + 4 = 0$
c) $x^2 - 5x + 0 = 0$

d) $x^2 - 5x - 6 = 0$

> **ℹ Note**
>
> What do you notice about the solutions as the constant term changes?

Application & Extension

Break Even

Real-world quadratic application

A company's profit function[1] $P = -2x^2 + 120x - 1600$

Find break-even points.

- Break-even: Solve $-2x^2 + 120x - 1600 = 0$
- Divide by –2: $x^2 - 60x + 800 = 0$
- Using formula: $x = \frac{60 \pm \sqrt{3600-3200}}{2} = \frac{60 \pm 20}{2}$
- Break-even at $x = 20$ or $x = 40$ (2,000 or 4,000 units)

Collaborative Problem-Solving

Market Analysis Challenge

Work in groups

A new product's market share M after t months follows:

$$M = -2t^2 + 12t$$

a) Find when market share is zero (factor completely)
b) Graph the market lifecycle

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> **ℹ Note**
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> We'll explore finding the maximum profit point when we study quadratic functions in Section 03.

Coffee Break - 15 Minutes

Final Assessment

5 minutes - individual work

Solve using the most efficient method:

[1] Where x = units sold (in hundreds).

a) $x^2 - 8x + 15 = 0$

b) $3x^2 + 2x - 1 = 0$

c) $x^4 - 10x^2 + 9 = 0$

Wrap-up & Synthesis

Key Takeaways

Essential skills mastered today

- Zero Product Property is fundamental to all equation solving
- Three methods for quadratics - each has its place
- Discriminant predicts solution behavior
- Biquadratic equations use substitution strategy
- Business applications often involve quadratic models

Next Session Preview

Session 02-04: Fractional, Radical & Cubic Equations

- Solving rational equations
- Domain restrictions
- Asymptotic behavior
- Business applications with rates