Session 02-02 - Systems of Linear Equations

Section 02: Equations & Problem-Solving Strategies

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Entry Quiz

Entry Quiz from Session 02-01

Individual work, then class review

1. Translate and solve: "The cost of 5 items plus a $30 delivery fee equals $180"
2. Break-even: A company has fixed costs of $4,000 and variable costs of $15 per unit. If they sell for $35 per unit, find the break-even quantity.
3. Mixture: How much 70% solution must be mixed with 20 liters of 30% solution to get a 45% solution?
4. Motion: Two cars start 450 km apart and drive toward each other. Car A travels 80 km/h, Car B travels 70 km/h. When do they meet?

Homework Presentations

Solutions Showcase

20 minutes - presentations and discussion

- Discuss your most challenging problem from Tasks 02-01
- Share your problem-solving approach
- Show potential alternative methods
- Ask questions about problems you found difficult

Key Concept

Systems of Equations

From Single to Multiple Unknowns

Previously, we mostly solved for one unknown:

$$ax + b = c$$

Now we tackle multiple unknowns simultaneously:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

. . .

Question: When might a business problem require multiple unknowns?

Warm-up: Simple System

5 minutes - collaborative task

Solve this system using any method you know:

$$
\begin{aligned}
x + y &= 10 \\
2x - y &= 5
\end{aligned}
$$

. . .

> **ℹ Note**
>
> How did you proceed?

Break - 10 Minutes

Solution Methods for 2×2 Systems

The Substitution Method

When to Use Substitution

One variable is already isolated or one coefficient is 1 or –1

1. Isolate one variable in one equation
2. Substitute into the other equation
3. Solve for the remaining variable
4. Back-substitute to find the first variable
5. Verify in both original equations

Example: Equilibrium

A market has a demand: $Q_d = 100 - 2P$ and a supply $Q_s = 20 + 3P$.

Find the equilibrium price and quantity.

Setup

At equilibrium: $Q_d = Q_s$

$$
\begin{aligned}
Q &= 100 - 2P \\
Q &= 20 + 3P
\end{aligned}
$$

Solution

Since both equal Q:

- $100 - 2P = 20 + 3P$

- $80 = 5P$
- $P = 16$
- $Q = 100 - 2(16) = 68$

Verification

- Demand: $Q_d = 100 - 2(16) = 68$ ✓
- Supply: $Q_s = 20 + 3(16) = 68$ ✓
- Equilibrium: Price = $16, Quantity = 68 units

The Elimination Method

When to Use Elimination

Best when no variable is easily isolated or the system is symmetric.

1. Align equations vertically
2. Multiply to create opposite coefficients
3. Add/Subtract to eliminate one variable
4. Solve for remaining variable
5. Back-substitute and verify

Example: Production Planning

A factory produces tables (T) and chairs (C) under constraints.

- Labor: 3T + 2C = 36 hours
- Materials: 2T + 2C = 28 units

How many of each can be produced?

Setup

$$3T + 2C = 36$$
$$2T + 2C = 28$$

Elimination

Subtract equation 2 from equation 1:

- $(3T + 2C) - (2T + 2C) = 36 - 28$
- $T = 8$

Substitution

Substitute into equation 2:

- $2(8) + 2C = 28$
- $2C = 12 \rightarrow C = 6$

Interpretation

- Produce 8 tables

- Produce 6 chairs
- Uses all available resources

Graphical Interpretation

Three Possible Outcomes

No Solution

- Parallel lines
- Inconsistent system
- Same slope, different intercepts

Infinite Solutions

- Same line
- Dependent equations
- One equation is multiple of other

Unique Solution

- Lines intersect once
- Most common case

. . .

Question: Can anyone here sketch these versions?

Spot the Pattern!

Work in groups to answer the following

Classify each system without solving:

1. $\{2x + 3y = 6$
   $4x + 6y = 12$
2. $\{2x + 3y = 6$
   $4x + 6y = 15$
3. $\{2x + 3y = 6$
   $3x + 2y = 6$

Extending to 3×3 Systems

Three Variables, Three Equations

The Challenge

With three unknowns, we have three independent equations, which requires care! We need to try eliminate systematically:

1. Use one equation to eliminate a variable from the other equations
2. Solve resulting 2×2 system for the other two variables
3. Back-substitute to find the eliminated variable from 1.

...

> Gaussian Elimination
>
> For systems with three or more variables, we could also use Gaussian Elimination. It's a systematic method using matrices and scales to any size. But as it is not required for the FSP, we decided to skip it here.

Guided Practice

Practice Set: 2×2 Systems

15 minutes - Individual then group work

Problem 1 (xx): Find the unit costs

- 5 units of A and 3 units of B cost $410
- 2 units of A and 4 units of B cost $320

Problem 2 (xxx): Find new equilibrium

- Demand: Q = 120 - 2P
- Supply: Q = 3P - 30
- $5 tax per unit on suppliers

Coffee Break - 15 Minutes

Collaborative Problem-Solving

Business Challenge

20 minutes - Work in groups

GlobalTrade operates in three regions with interconnected pricing:

Market conditions:

- Region A: Price affects demand in all regions
- Region B: Competes directly with Region C
- Region C: Premium market

Help Global Trade

The relationships

1. $P_A + 0.5P_B + 0.3P_C = 100$ (Combined market index)
2. $2P_A - P_B + P_C = 80$ (Competitive balance)
3. $P_A + P_B - 2P_C = -40$ (Premium differential)

a) Find equilibrium prices
b) How do prices change if Region B does not exit?

Method Selection Strategy

Decision Framework

2×2 Systems

Use Substitution when:

- One equation solved for a variable
- Coefficients are 1 or –1
- Word problems with clear relationships

Use Elimination when:

- Similar-looking equations
- Integer coefficients and/or symmetric systems

3×3 Systems

Use Systematic Elimination and Substitution when:

- Clean integer coefficients
- Hand calculation

Use Gaussian Elimination when:

- Need systematic approach
- Very complicated numbers
- But not neccessary for FSP! :)

Quick Decision Practice

Think individually, then discuss

Which method would you choose?

1. $\{y = 3x - 5$
   $2x + y = 10$
2. $\{3x + 4y = 25$
   $5x + 4y = 35$
3. $\{2x + 3y - z = 7$
   $y + 2z = 5$
   $z = 3$

Wrap-up & Synthesis

Key Takeaways

- Two methods for 2×2: Substitution vs. Elimination
- Three outcomes possible: Unique, none, or infinite solutions
- Business insights: Inconsistent constraints reveal planning issues
- Method selection matters: Choose based on equation structure

. . .

> ⚠ Watch Out For These!
>
> 1. Arithmetic errors in elimination - Check each step
> 2. Forgetting to verify solutions - Always substitute back
> 3. Missing special cases - Check for parallel/identical lines
> 4. Rounding too early - Keep fractions until the end

Final Assessment

10 minutes - Individual work

A company produces products A and B:

- Combined production: 50 units
- Revenue: $30A + $40B = $1,700
- Labor hours: 2A + 3B = 120

a) Set up the system of equations
b) Identify any redundancy in the constraints
c) Solve for A and B
d) Calculate total profit if costs are $20 per unit for both

Next Session Preview

Session 02-03: Quadratic & Biquadratic Equations

We'll explore:

- The quadratic formula and discriminant
- Completing the square
- Projectile motion and optimization
- Biquadratic substitution techniques

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> Preparation Tip
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> Review factoring from Section 01 - we'll apply it to quadratics!