# Session 01-05 - Logarithms & Substitution

## Section 01: Mathematical Foundations & Algebra

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# Entry Quiz

## Quick Review from Last Session

Complete individually, then we discuss

a) Factor completely: $x^3 - 27$

b) Simplify: $\sqrt{48} + \sqrt{12} - \sqrt{75}$

c) Rationalize: $\frac{3}{\sqrt{5}-2}$

d) Factor using AC method: $2x^2 + 7x + 3$

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> Tip
>
> Let's review together!

# Student Presentations

## Homework Showcase

20 minutes for presentations and discussion

- Present and discuss your solutions from Tasks 01-04
- Share any challenging problems or interesting approaches
- Use this time to clarify concepts before we move forward

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> Note
>
> Today we build on factorization and radicals with new powerful tools!

# Substitution for Factorization

## What is Substitution for Factorization?

Making complex expressions simpler by introducing a new variable

Sometimes factorization becomes easier when we substitute part of an expression with a simpler variable.

- Strategy: Replace a repeated expression with a single variable
- Factor the simpler expression
- Substitute back to get the final answer
- Why it works: Reduces cognitive load and reveals hidden patterns

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> Tip
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> Look for expressions that appear multiple times or have a clear “inner” structure!

## When to Use Substitution

Recognize these common patterns

- Quadratic in form: $(x^2)^2 + 5(x^2) + 6$
- Repeated expressions: $(2x + 1)^2 - 3(2x + 1) - 10$
- Complex nested terms: $\sqrt{x+1} - 2\sqrt{x+1} + 1$
- Trigonometric expressions: $\sin^2(x) + 3\sin(x) + 2$

. . .

> Note
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> The key is identifying what to substitute - look for the “building block” that repeats!

## Substitution Examples

Let’s work through some examples step by step

## Example A

Factor $x^4 - 13x^2 + 36$

- Step 1: Let $u = x^2$, so $x^4 = u^2$
- Step 2: Substitute: $u^2 - 13u + 36$
- Step 3: Factor: $(u - 4)(u - 9)$
- Step 4: Substitute back: $(x^2 - 4)(x^2 - 9)$
- Step 5: Factor completely: $(x - 2)(x + 2)(x - 3)(x + 3)$

## Example B

Factor $x + 6\sqrt{x} + 8$

- Step 1: Notice this involves $x$ and $\sqrt{x}$, where $x = (\sqrt{x})^2$
- Step 2: Let $u = \sqrt{x}$, so $x = u^2$
- Step 3: Substitute: $u^2 + 6u + 8$
- Step 4: Factor: $(u + 2)(u + 4)$
- Step 5: Substitute back: $(\sqrt{x} + 2)(\sqrt{x} + 4)$

## Example C

Factor $3x^6 - 11x^3 - 20$

- Step 1: Let $u = x^3$, so $x^6 = u^2$ and we have $3u^2 - 11u - 20$
- Step 2: Use AC method: $ac = 3(-20) = -60$
- Step 3: Find factors of –60 that sum to –11: (4, –15)
- Step 4: Rewrite: $3u^2 + 4u - 15u - 20$
- Step 5: Group: $u(3u + 4) - 5(3u + 4) = (u - 5)(3u + 4)$
- Step 6: Substitute back: $(x^3 - 5)(3x^3 + 4)$

## Common Substitutions

Simplification tricks

When you see:

- $3^{2x} \longrightarrow$ Let $u = 3^x$, then $3^{2x} = u^2$
- $\sqrt{x}$ appearing multiple times $\longrightarrow$ Let $u = \sqrt{x}$
- Symmetric expressions $\longrightarrow$ Look for factoring patterns
- Repeating decimals $\longrightarrow$ Use algebraic method to find fraction

## Practice with Substitution

Try these on your own

Work individually, then we'll discuss solutions:

a) Factor: $x^6 + 8x^3 + 16$

b) Factor: $(\sqrt{x} - 2)^2 - 5(\sqrt{x} - 2) + 6$

c) Factor: $16x^4 - 81$

d) Factor: $(x^2 + 3x)^2 - 8(x^2 + 3x) + 15$

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> ! Important
>
> Always check if you can factor further after substituting back!

### More Advanced Substitution

Sometimes you need to think a little bit more

Example: Factor $x^{2/3} - 5x^{1/3} + 6$

- Observation: This involves fractional exponents
- Key insight: Let $u = x^{1/3}$, so $x^{2/3} = \left(x^{1/3}\right)^2 = u^2$
- Step 1: Substitute: $u^2 - 5u + 6$
- Step 2: Factor: $(u-2)(u-3)$
- Step 3: Substitute back: $\left(x^{1/3} - 2\right)\left(x^{1/3} - 3\right)$
- Step 4: Can also write as: $(\sqrt[3]{x} - 2)(\sqrt[3]{x} - 3)$

# Break - 10 Minutes

# Logarithms - The Basics

### What is a Logarithm?

The logarithm is the inverse of exponentiation

$$\text{If} \quad a^x = b \text{ , then } \log_a(b) = x$$

Think of it as: "What power do I raise $a$ to get $b$?"

- $2^3 = 8$ means $\log_2(8) = 3$
- $5^x = 125$ means $x = \log_5(125) = 3$

. . .

i Note

Standard notation:

- $\log$ without a base means $\log_{10}$ (common logarithm)
- $\ln$ means $\log_e$ where $e \approx 2.718$ (natural logarithm)

### Key Logarithm Properties

These follow directly from exponent laws!

| Property | Formula | Why it works |
|---|---|---|
| $\log_a(1) = 0$ | Because $a^0 = 1$ | Any base to the 0 is 1 |
| $\log_a(a) = 1$ | Because $a^1 = a$ | Base to the 1st is itself |
| $\log_a(a^x) = x$ | Direct from definition | Inverse operations |
| $a^{\log_a(x)} = x$ | Direct from definition | Inverse operations |

> **Note**
>
> Important: Logarithms are transcendental functions - they cannot be expressed using only algebraic operations (unlike polynomials, radicals, and rational functions).

### Laws of Logarithms

These transform complex operations into simple ones

| Rule | Formula | Example |
|---|---|---|
| Product | $\log_a(xy) = \log_a(x) + \log_a(y)$ | $\log(20) = \log(4) + \log(5)$ |
| Quotient | $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ | $\log\left(\frac{100}{4}\right) = \log(100) - \log(4)$ |
| Power | $\log_a(x^n) = n\log_a(x)$ | $\log(8^3) = 3\log(8)$ |

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> **Common Mistake**
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> $\log(x + y) \neq \log(x) + \log(y)$
>
> There's NO simple rule for $\log(x + y)$!

## Working with Logarithms

### Evaluating Logarithms

### Example A

Find $\log_3(81)$

- Ask: "3 to what power equals 81?"
- $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$
- Therefore: $\log_3(81) = 4$

### Example B

Simplify $\log_2(32) + \log_2(8) - \log_2(4)$

- Method 1: Evaluate each
  - $\log_2(32) = 5$, $\log_2(8) = 3$, $\log_2(4) = 2$
  - Result: $5 + 3 - 2 = 6$
- Method 2: Use laws
  - $= \log_2\left(\frac{32 \times 8}{4}\right) = \log_2(64) = 6$

### Change of Base Formula

Convert between different bases

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} = \frac{\ln(x)}{\ln(a)} = \frac{\log(x)}{\log(a)}$$

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Example: Find $\log_5(30)$

- $\log_5(30) = \frac{\ln(30)}{\ln(5)} = \frac{3.401}{1.609} \approx 2.113$
- Check: $5^{2.113} \approx 30$ ✓

### Individual Exercise 01

Practice logarithm skills

a) Evaluate: $\log_4(64)$

b) Simplify: $\log_3(9) + \log_3(27)$

c) Solve: $\log_5(x + 4) = 2$

d) Express as a single logarithm: $2\log(x) - \log(y) + \log(3)$

# Logarithms in the Real World

## Why Logarithms Matter

From protecting your hearing to predicting disasters

- The Challenge: Natural phenomena span enormous ranges
- Human perception: We sense changes proportionally, not linearly
- The Solution: Logarithmic scales compress huge ranges into manageable numbers
- Real Impact: These scales help save lives and advance science

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> **i Note**
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> Historical Note: Logarithms were invented in 1614 by John Napier to simplify astronomical calculations. Today, they're essential for measuring everything from sound to earthquakes!

## Scientific Applications

## Sound Intensity

Why we need the decibel scale: Sound intensity ranges from $10^{-12}$ to $10^{12}$ watts/m$^2$ - that's 24 orders of magnitude!

Decibel formula: $L = 10\log\left(\frac{I}{I_0}\right)$ dB

- Whisper: 30 dB (1,000× threshold)
- Normal conversation: 60 dB (1,000,000× threshold)
- Rock concert: 110 dB (100,000,000,000× threshold)

- Jet engine: 140 dB (causes immediate hearing damage!)

> ⚠ Warning
>
> Health Alert: Each 10 dB increase = 10× intensity. That rock concert isn't just "a bit louder" - it's 1,000× more intense than conversation!

### Earthquake Magnitude

The problem with linear scales: Earthquake energy ranges from equivalents of small explosions to thousands of atomic bombs!

Richter scale: $M = \log_{10}\left(\frac{A}{A_0}\right)$

- Magnitude 3: Barely felt (like a large truck passing)
- Magnitude 5: Light damage (100× stronger than Mag 3)
- Magnitude 7: Major earthquake (10,000× stronger than Mag 3)
- Magnitude 9: Great earthquake (1,000,000× stronger than Mag 3)

### Financial Applications

Why Natural Logarithm for Finance?

The connection to continuous growth

- Any logarithm works: $t = \frac{\log_{10}(2)}{\log_{10}(1+r)} = \frac{\ln(2)}{\ln(1+r)}$
- But ln is natural because it connects to continuous compounding
- Continuous compounding formula: $A = Pe^{rt}$ (where $e \approx 2.718$)
- Why e appears: It's the limit as compounding frequency → infinity

### Compound Interest Time Calculations

How long to double your money?

- Formula: $2P = P(1+r)^t$
- Simplify: $2 = (1+r)^t$
- Take logarithms: $\ln(2) = t \cdot \ln(1+r)$
- Solve: $t = \frac{\ln(2)}{\ln(1+r)}$

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> 💡 Tip
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> Rule of 72: At r% interest, doubling time ≈ $\frac{72}{r}$ years

# Coffee Break - 15 Minutes

# Advanced Algebraic Techniques

## Solving Exponential Equations

Logarithms are the key tool

### Example A

Solve $3^{2x-1} = 81$

- Recognize: $81 = 3^4$
- So: $3^{2x-1} = 3^4$
- Therefore: $2x - 1 = 4$
- Solve: $x = 2.5$

### Example B

Solve $5^x = 30$

- Take logarithms: $\log(5^x) = \log(30)$
- Use power rule: $x \cdot \log(5) = \log(30)$
- Solve: $x = \frac{\log(30)}{\log(5)} \approx 2.113$

# Expanding Binomial Powers

## Pascal's Triangle

A pattern of binomial coefficients

```
Row 0:              1
Row 1:            1   1
Row 2:          1   2   1
Row 3:        1   3   3   1
Row 4:      1   4   6   4   1
Row 5:    1   5   10  10  5   1
Row 6:  1   6   15  20  15  6   1
```

- Each number = sum of two above
- Row n gives coefficients for $(a + b)^n$
- Symmetric pattern

## Pair Exercise

Work together on binomial problems

a) Expand completely: $(x - 3)^3$

## Individual Exercise

Try to solve the following individually

a) If $\log_2(x) + \log_4(x) = 3$, find x.
b) Expand: $(3x - 2y)^3$
c) Simplify: $\log_3(27) - \log_3(3)$
d) Solve: $2^{x+1} = 32$

# Wrap-up

## Key Takeaways

- Substitution is a powerful technique for simplifying expressions
- Logarithms are inverse exponentials
- The logarithm laws simplify complex calculations
- Pascal's triangle gives binomial coefficients
- These tools are essential for calculus, statistics, and finance

## For Next Time

Homework: Complete Tasks 01-05

Preview of Session 01-06 (Synthesis):

- Integration of ALL Section 1 concepts
- Complex problem-solving strategies
- Business case studies

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> ! Important
>
> Start reviewing all Section 1 material - synthesis session next!

# Questions & Discussion

## Discussion

Your questions and insights are welcome!

- Clarifications on logarithms?
- Connections to other mathematical topics?
- Applications you're curious about?

## See you next session!

The synthesis session will bring everything together!