Session 01-04 - Advanced Factorization & Radicals

Section 01: Mathematical Foundations & Algebra

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Entry Quiz

Quick Review from Last Session

Complete individually, then we review as group

a) Simplify: $\frac{(2x^3)^2 \cdot x^{-5}}{4x^2}$

b) Factor: $9x^2 - 25$

c) Solve: $|2x - 4| > 6$

d) Express in scientific notation: 0.0000234×10^3

e) Factor: $3x^2 - 12$

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> Tip
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> Ok, lets talk about your solutions together!

Student Presentations

Homework Showcase

20 minutes for discussing your solutions

- Present and discuss your solutions from Tasks 01-03
- Focus on the most challenging aspects or tasks
- Share any challenging aspects or alternative approaches

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> Note
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> Today we build on basic factorization with advanced techniques and introduce radicals!

Advanced Factorization

Repetition: What is Factorization?

Breaking expressions into products of simpler factors

Factorization means writing an expression as a product of its factors.

- Example: $12 = 3 \times 4$ (factoring numbers)
- Algebra: $x^2 + 5x + 6 = (x + 2)(x + 3)$ (factoring polynomials)
- Reverse of expansion: $(x + 2)(x + 3) \rightarrow x^2 + 5x + 6$

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i Why Factor?

- Because the exam requires you to know this!
- Can help us to solve equations later
- Cancel common factors in fractions and help to simplify expressions

Factoring Quadratics: $ax^2 + bx + c$

When $a = 1$: Find factors of c that sum to b

Example A

Factor $x^2 + 7x + 12$

- Need factors of 12 that add to 7
- Pairs: (1,12), (2,6), (3,4)
- Check: $3 + 4 = 7$ ✓
- Result: $(x + 3)(x + 4)$

Example B

Factor $x^2 - 5x - 14$

- Need factors of −14 that add to −5
- Pairs: (-1,14), (1,-14), (-2,7), (2,-7)
- Check: $2 + (-7) = -5$ ✓
- Result: $(x + 2)(x - 7)$

The AC Method for $ax^2 + bx + c$

When the leading coefficient $a \neq 1$

Factor: $6x^2 + 13x + 5$

- Step 1: Find $ac = 6 \times 5 = 30$
- Step 2: Find factors of 30 that sum to 13, e.g. (3,10)
- Step 3: Rewrite: $6x^2 + 3x + 10x + 5$
- Step 4: Group: $3x(2x + 1) + 5(2x + 1)$

- Step 5: Factor: $(3x + 5)(2x + 1)$

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> Tip
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> Check your solution by expanding: $(3x + 5)(2x + 1) = 6x^2 + 13x + 5$ ✓

The AC Method When $ac < 0$

When ac is negative, factors have opposite signs

Factor: $6x^2 + 7x - 5$

- Step 1: Find $ac = 6 \times (-5) = -30$ (negative!)
- Step 2: Find factors of −30 that sum to 7, e.g. (10, −3)
  - Need one positive, one negative factor!
- Step 3: Rewrite: $6x^2 + 10x - 3x - 5$
- Step 4: Group: $2x(3x + 5) - 1(3x + 5)$
- Step 5: Factor: $(2x - 1)(3x + 5)$

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> ! Important
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> When $ac < 0$: Look for factor pairs with opposite signs that sum to b

The Discriminant: Can We Factor?

A mathematical detective tool that saves us time!

The Problem: You're given $3x^2 + 7x + 11$ and asked to factor it. Do you: - Spend 10 minutes trying every possible combination? - Or spend 30 seconds checking if it's even possible?

The Solution: The discriminant! It's like a "factorability test" that tells us instantly whether we're wasting our time.

What is a Discriminant? The discriminant $\Delta = b^2 - 4ac$ is the expression under the square root in the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

How to Calculate $\boldsymbol{b^2 - 4ac}$:

- Step 1: Identify a, b, c from $ax^2 + bx + c$
- Step 2: Calculate $b^2 - 4ac$
- Step 3: Check if result is a perfect square

Perfect Squares and Factoring

Perfect squares are the "clean" numbers of mathematics

Historical Note: Ancient Greeks called these "square numbers" because you can arrange them into perfect square arrays of dots!

Think of perfect squares like this: They're numbers whose square roots are "nice" whole numbers - no messy decimals!

Perfect squares: $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, $25 = 5^2$, $36 = 6^2$, etc.

Non-perfect squares: 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21…

> Tip
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> Quick Check: To see if a number is a perfect square, find its square root. If the square root is a whole number, it's a perfect square!
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> Example: $\sqrt{25} = 5$ (whole number) $\longrightarrow$ perfect square! But $\sqrt{21} \approx 4.58...$ (messy decimal) $\longrightarrow$ not perfect!
>
> Fun Fact: The largest perfect square less than 100 is 81. Can you name the next one?

The AC Method Rule

The ultimate time-saver: Know before you try!

The Golden Rule: A quadratic $ax^2 + bx + c$ can be factored over the integers if and only if the discriminant $\Delta = b^2 - 4ac$ is a perfect square.

Real-World Application: Imagine you're an engineer designing a bridge. You need to solve $2x^2 + 13x + 15 = 0$. Should you factor or use the quadratic formula?

- Check: $\Delta = 13^2 - 4(2)(15) = 169 - 120 = 49 = 7^2$ ✓
- Since it's a perfect square, factoring will work: $(2x + 3)(x + 5)$
- Time saved: 5 minutes of trial and error!

Counter-example: For $x^2 + 3x - 3$: $\Delta = 9 + 12 = 21$ (not perfect) $\longrightarrow$ Don't waste time trying to factor!

> Note
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> Pro Tip: Always check the discriminant first! It's like checking if a door is locked before trying to open it.

Factoring by Grouping

Group terms with common factors

Example A

Factor $x^3 + 2x^2 - 3x - 6$

- Group: $(x^3 + 2x^2) + (-3x - 6)$
- Factor each: $x^2(x + 2) - 3(x + 2)$

- Common factor: $(x+2)(x^2-3)$

Example B

Factor $2x^3 - x^2 - 8x + 4$

- Group: $(2x^3 - x^2) + (-8x + 4)$
- Factor: $x^2(2x-1) - 4(2x-1)$
- Result: $(2x-1)(x^2-4)$
- Even further: $(2x-1)(x+2)(x-2)$

Sum and Difference of Cubes

These patterns are worth memorizing!

Pattern	Formula
Sum of Cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of Cubes	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Examples:

- $x^3 + 8 = x^3 + 2^3 = (x+2)(x^2 - 2x + 4)$
- $x^3 - 27 = x^3 - 3^3 = (x-3)(x^2 + 3x + 9)$
- $8x^3 + 125 = (2x)^3 + 5^3 = (2x+5)(4x^2 - 10x + 25)$

Individual Exercise 01

Go ahead and factor these completely

a) $3x^2 + 10x + 8$

b) $x^3 - 64$

c) $2x^3 + 3x^2 - 8x - 12$

d) $4x^2 - 11x - 3$

e) $27x^3 + 8$

Break - 10 Minutes

Roots and Radicals

Understanding Roots

Roots ask: "What number gives me this when raised to a power?"

- $\sqrt{25} = 5$ because $5^2 = 25$
- $\sqrt[3]{8} = 2$ because $2^3 = 8$
- $\sqrt[4]{81} = 3$ because $3^4 = 81$

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ℹ The Sign Rules

- Square roots (and even roots): Always positive by convention
  - $\sqrt{9} = 3$ (not -3, even though $(-3)^2 = 9$)
- Cube roots (and odd roots): Keep the original sign
  - $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$

Properties of Radicals

These properties allow us to simplify

Property	Formula	Example
Product	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$
Quotient	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt{\frac{16}{4}} = \frac{\sqrt{16}}{\sqrt{4}} = 2$
Power	$\sqrt[n]{a^m} = a^{m/n}$	$\sqrt[3]{x^6} = x^2$

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Tip

Key idea: Look for perfect squares, cubes, etc. that you can "pull out" of the radical!

Simplifying Radicals

Strategy: Extract perfect powers from under the radical

Example A

Simplify $\sqrt{72}$

- Factor: $72 = 36 \times 2 = 6^2 \times 2$
- Extract: $\sqrt{36 \times 2} = \sqrt{36} \times \sqrt{2}$
- Result: $6\sqrt{2}$

Example B

Simplify $\sqrt{50x^5y^3}$

- Factor: $50 = 25 \times 2$, $x^5 = x^4 \cdot x$, $y^3 = y^2 \cdot y$
- Extract: $\sqrt{25x^4y^2 \cdot 2xy}$
- Result: $5x^2y\sqrt{2xy}$

Operations with Radicals

Can only combine like radicals!

Example: Simplify $3\sqrt{12} + 2\sqrt{27} - \sqrt{48}$

- Simplify each term:

- $3\sqrt{12} = 3 \cdot 2\sqrt{3} = 6\sqrt{3}$
  - $2\sqrt{27} = 2 \cdot 3\sqrt{3} = 6\sqrt{3}$
  - $\sqrt{48} = 4\sqrt{3}$
- Combine: $6\sqrt{3} + 6\sqrt{3} - 4\sqrt{3} = 8\sqrt{3}$

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Tip

Always simplify radicals first before combining!

Rationalizing Denominators

What is Rationalizing?

Removing radicals from denominators

- Rationalize means to rewrite a fraction
- We want no square roots (or other radicals) in the denominator.

- Before: $\frac{1}{\sqrt{2}}$ (radical in denominator)
- After: $\frac{\sqrt{2}}{2}$ (no radical in denominator)

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Why Rationalize?

Easier calculation before calculators, standard form for mathematical expressions, and often simplifies further operations.

Simple Radical Denominators

Basic principle: Multiply by a form of 1 that eliminates the radical

Example A

Rationalize $\frac{3}{\sqrt{5}}$

- Multiply by $\frac{\sqrt{5}}{\sqrt{5}}$
- $\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$

Example B

Rationalize $\frac{3}{\sqrt[3]{2}}$

- We could make the denominator a perfect cube
- Multiply by $\frac{\sqrt[3]{4}}{\sqrt[3]{4}}$ (since $2 \times 4 = 8 = 2^3$)
- Result: $\frac{3\sqrt[3]{4}}{2}$

Using Conjugates

A conjugate flips the sign between terms

Definition: The conjugate of $a + b\sqrt{c}$ is $a - b\sqrt{c}$

- Example: Conjugate of $\sqrt{3} + 1$ is $\sqrt{3} - 1$
- Key property: $(a + b)(a - b) = a^2 - b^2$ (difference of squares)
- Why it works: The radical terms cancel out when multiplied!

Conjugates

Use the conjugate to eliminate radicals

Example: Rationalize $\frac{2}{\sqrt{3}+1}$

- Multiply by conjugate: $\frac{\sqrt{3}-1}{\sqrt{3}-1}$
- Numerator: $2\left(\sqrt{3} - 1\right) = 2\sqrt{3} - 2$
- Denominator: $\left(\sqrt{3}\right)^2 - 1^2 = 3 - 1 = 2$
- Result: $\frac{2\sqrt{3}-2}{2} = \sqrt{3} - 1$

Pair Exercise 01

Work together on these problems

a) Simplify: $\sqrt{72} + \sqrt{50} - \sqrt{98}$

b) Rationalize: $\frac{4}{\sqrt{6}-\sqrt{2}}$

c) Simplify: $\sqrt[3]{54x^7y^5}$

d) Simplify: $\frac{\sqrt{45x^3}}{\sqrt{5x}}$

Coffee Break - 15 Minutes

Complex Algebraic Manipulation

Combining All Techniques

Use factorization, exponents, and radicals together

Example: Simplify $\frac{x^2-4}{x^2-x-6} \cdot \frac{x^2-9}{x+2}$

- Factor everything:
  - $x^2 - 4 = (x + 2)(x - 2)$
  - $x^2 - x - 6 = (x - 3)(x + 2)$
  - $x^2 - 9 = (x + 3)(x - 3)$
- Rewrite: $\frac{(x+2)(x-2)}{(x-3)(x+2)} \cdot \frac{(x+3)(x-3)}{x+2}$
- Result: $\frac{(x-2)(x+3)}{x+2}$

Complex Fractions

Simplify: $\frac{\frac{\frac{x}{3}}{2}}{x}$

- Remember: Dividing by a fraction means multiply by its reciprocal
- $\frac{\frac{x}{3}}{\frac{2}{x}} = \frac{x}{3} \times \frac{x}{2}$
- Multiply: $\frac{x \times x}{3 \times 2} = \frac{x^2}{6}$

Practice

Individual Exercise 02

Apply all techniques together

a) Factor completely: $8x^3 - 125$

b) Simplify: $\sqrt{75x^3} - x\sqrt{12x} + 2\sqrt{27x^3}$

c) Rationalize: $\frac{3}{2-\sqrt{3}}$

d) Simplify: $\frac{x^3-8}{x^2-4} \div \frac{x^2+2x+4}{x+2}$

Wrap-up

Key Takeaways

- AC method handles quadratics with $a \neq 1$
- Grouping wcan work for four-term polynomials
- Cube formulas follow specific patterns
- Radicals simplify by extracting perfect powers
- Rationalization uses conjugates for binomials

For Next Time

Homework: Complete Tasks 01-04

Preview of Session 01-05:

- Logarithms and their properties
- Binomial theorem and Pascal's triangle
- Advanced algebraic applications

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> ! Important
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> Practice factorization and radicals - they're essential for all advanced math!

Questions & Discussion

Open Floor

Your questions and insights are welcome!

- Which factorization method is most challenging?

- Any confusion about radicals or rationalization?
- Real-world applications you're curious about?

See you next session!

Keep practicing - mastery comes through repetition!