Session 01-03 - Core Algebra & Exponents

Section 01: Mathematical Foundations & Algebra

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Entry Quiz

Quick Review from Last Session

Complete individually, then we review the results together

a) Express in set-builder notation: All odd integers less than 20
b) If $A = \{1, 3, 5, 7\}$ and $B = \{3, 4, 5, 6\}$, find $A \cap B$ and $A \cup B$
c) Is $0.\overline{36}$ rational? If yes, express as a fraction.
d) True or false: If $p \Rightarrow q$ is true and q is false, what can we say about p?

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> Tip
>
> Ok, let's review together!

Homework Presentations

Homework Showcase

20 minutes for presentations and discussion

- Present and discuss your solutions from Tasks 01-02
- Share any challenging problems or interesting approaches
- This is your opportunity to ask questions and learn from each other

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> Tip
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> Remember: Explaining your solution helps solidify your understanding!

Algebraic Expressions

"Algebraic"?

An algebraic expression combines:

- Variables: letters representing unknown values (x, y, z, …)
- Constants: fixed numbers (2, π, −5, …)
- Operations: +, −, ×, ÷, and exponents
- Example: $x^3 - 2x^2 + 7 \ldots$

> ⚠ Warning
>
> Not algebraic: $\sin(x)$, $\log(x)$, e^x
>
> These are transcendental functions - but no need to worry about this for now!

Order of Operations - PEMDAS

1. Parentheses (brackets, braces)
2. Exponents (powers, roots)
3. Multiplication and Division (left to right)
4. Addition and Subtraction (left to right)

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Example: $2 + 3 \times 4^2 - (5 - 3) \div 2$

- Step 1 (Parentheses): $(5 - 3) = 2$
- Step 2 (Exponents): $4^2 = 16$
- Step 3 (Multiply/Divide): $3 \times 16 = 48$ and $2 \div 2 = 1$
- Step 4 (Add/Subtract): $2 + 48 - 1 = 49$

Practice PEMDAS Together

Let's work through this step-by-step

Evaluate: $\frac{3^2+2\times(4-1)}{5-2}$

- Numerator first:
  - Parentheses: $(4 - 1) = 3$
  - Exponent: $3^2 = 9$
  - Multiply: $2 \times 3 = 6$
  - Add: $9 + 6 = 15$
- Denominator: $5 - 2 = 3$
- Final division: $\frac{15}{3} = 5$

Individual Exercise 01

Practice order of operations for yourself

Evaluate and then we'll review together:

a) $4 + 2^3 \times 3 - 12 \div 4$
b) $(3 + 2)^2 - 3 \times (7 - 4)$
c) $\frac{2^3+3\times2}{10-3}$
d) $5 \times [2 + 3 \times (4 - 2)^2]$

Break - 10 Minutes

Laws of Exponents

The Fundamental Rules

These laws help us manage exponents!

Rule	Formula	Example
Product Rule	$a^m \cdot a^n = a^{m+n}$	$x^3 \cdot x^4 = x^7$
Quotient Rule	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{x^5}{x^2} = x^3$
Power Rule	$(a^m)^n = a^{mn}$	$(x^3)^2 = x^6$
Product Power	$(ab)^n = a^n b^n$	$(2x)^3 = 8x^3$
Quotient Power	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{x}{3}\right)^2 = \frac{x^2}{9}$

Special Exponent Values

These are essential to memorize!

- $a^0 = 1$ (for any $a \neq 0$)
- $a^1 = a$
- $a^{-n} = \frac{1}{a^n}$
- $a^{1/n} = \sqrt[n]{a}$
- $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

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⚠ Common Mistake

$(x+y)^2 \neq x^2 + y^2$, remember: $(x+y)^2 = x^2 + 2xy + y^2$

More about this later!

Let's Practice Together

Simplify: $\frac{(3x^2y)^2 \cdot x^{-3}}{9xy^2}$

- Expand the power: $\frac{9x^4y^2 \cdot x^{-3}}{9xy^2}$
- Combine exponents in numerator: $\frac{9x^{4-3}y^2}{9xy^2}$
- Simplify: $\frac{9x^1y^2}{9xy^2}$
- Finalize: $\frac{1}{1} = 1$

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Your turn: Try $\frac{(2a^3)^2 \cdot a^{-4}}{4a}$

Pair Exercise 01

Work together on these exponent problems in pairs

Simplify completely:

a) $(x^3)^2 \cdot x^{-4}$
b) $\frac{12x^5y^3}{3x^2y}$
c) $\left(\frac{2x^2}{y}\right)^3 \cdot \frac{y^2}{4x^3}$
d) $(3^2)^3 \cdot 3^{-5}$

Scientific Notation

Why Scientific Notation?

Essential for extreme values!

Scientific notation: $a \times 10^n$ where $1 \leq |a| < 10$

Real-world examples:

- World population: $8,000,000,000 = 8.0 \times 10^9$ people
- Virus diameter: $0.0000001 = 1 \times 10^{-7}$ meters
- US National debt: $\$31,400,000,000,000 = 3.14 \times 10^{13}$ dollars

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> Tip
>
> Scientific notation makes calculations with very large or very small numbers practical!

Converting to Scientific Notation

Large Numbers

Convert 56,700,000

- Move decimal left to get one non-zero digit
- Count moves: 7 positions left
- Result: 5.67×10^7

Small Numbers

Convert 0.00000423

- Move decimal right to get one non-zero digit
- Count moves: 6 positions right
- Result: 4.23×10^{-6}

Key Rule

- Moving decimal left $\longrightarrow$ positive exponent
- Moving decimal right $\longrightarrow$ negative exponent

Operations with Scientific Notation

Multiplication: $(3 \times 10^5) \times (2 \times 10^3)$

- Multiply coefficients: $3 \times 2 = 6$
- Add exponents: $10^5 \times 10^3 = 10^8$
- Result: 6×10^8

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Division: $\frac{8.4 \times 10^7}{2.1 \times 10^4}$

- Divide coefficients: $8.4 \div 2.1 = 4$
- Subtract exponents: $10^7 \div 10^4 = 10^3$
- Result: 4×10^3

Coffee Break - 15 Minutes

Absolute Value

Understanding Absolute Value

The absolute value $|x|$ represents the distance from zero

Definition:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Examples:

- $|5| = 5$ (already positive)
- $|-3| = 3$ (make positive)
- $|0| = 0$ (zero stays zero)

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> **ℹ Note**
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> Think of absolute value as “removing the sign” or “distance without direction”

Properties of Absolute Value

These properties are fundamental for working with absolute values

- Non-negativity: $|x| \geq 0$ for all x (absolute value is never negative)
- Zero property: $|x| = 0$ if and only if $x = 0$

- Multiplicative: $|xy| = |x| \cdot |y|$
- Quotient: $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ (when $y \neq 0$)
- Squares: $|x|^2 = x^2$

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> Geometric Interpretation
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> $|x|$ as the distance from x to 0 on the number line.

Solving Absolute Value Equations

Example: Solve $|2x - 6| = 4$

Two cases to consider:

- Case 1: $2x - 6 = 4$
  - $2x = 10$
  - $x = 5$
- Case 2: $2x - 6 = -4$
  - $2x = 2$
  - $x = 1$
- Solution: $x \in \{1, 5\}$

Absolute Value Inequalities

Type 1: Less Than

$|x| < a$ means $-a < x < a$

Example: $|x - 3| < 2$

- $-2 < x - 3 < 2$
- $1 < x < 5$
- Solution: $(1, 5)$

Type 2: Greater Than

$|x| > a$ means $x < -a$ OR $x > a$

Example: $|x - 3| > 2$

- $x - 3 < -2$ OR $x - 3 > 2$
- $x < 1$ OR $x > 5$
- Solution: $(-\infty, 1) \cup (5, \infty)$

Application

Quality Control: Bolts must be 20 ± 0.3 mm

- Specification: $|d - 20| \leq 0.3$
- Acceptable range: $[19.7, 20.3]$ mm

Basic Factorization

Common Factor Method

Always check for common factors first!

Example: Factor $12x^3 - 18x^2 + 6x$

- Find the GCF (Greatest Common Factor):
  - Numbers: GCF of 12, 18, 6 is 6
  - Variables: lowest power of x is x^1
  - GCF = $6x$
- Factor out: $6x(2x^2 - 3x + 1)$

Difference of Squares

Pattern: $a^2 - b^2 = (a + b)(a - b)$

Examples:

- $x^2 - 9 = x^2 - 3^2 = (x + 3)(x - 3)$
- $4x^2 - 25 = (2x)^2 - 5^2 = (2x + 5)(2x - 5)$
- $16x^2 - 49y^2 = (4x)^2 - (7y)^2 = (4x + 7y)(4x - 7y)$

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> ⚠ Warning
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> $a^2 + b^2$ cannot be factored! This is because we need a difference (subtraction) between two perfect squares to use this factorization pattern. When we have a sum (addition) like $x^2 + 9$, there's no real number factorization.

Perfect Square Trinomials

Recognize these special patterns

Pattern	Formula	How to Recognize
$(a + b)^2$	$a^2 + 2ab + b^2$	Perfect squares, middle = 2×(product of roots)
$(a - b)^2$	$a^2 - 2ab + b^2$	Same, but middle term is negative

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Examples:

- $x^2 + 6x + 9 = (x + 3)^2$
- $x^2 - 10x + 25 = (x - 5)^2$
- $4x^2 + 12x + 9 = (2x + 3)^2$

Group Exercise 01

Practice factorization together (8 minutes)

Factor completely:

a) $3x^2 - 27$
b) $x^2 - 8x + 16$
c) $25x^2 - 9$
d) $5x^3 - 20x$
e) $x^2 + 14x + 49$

Business Applications

Compound Interest Revisited

The power of exponential growth

Formula: $A = P(1 + r)^t$

- P = Principal (initial amount)
- r = Interest rate (as decimal)
- t = Time periods
- A = Final amount

Example: €5,000 at 6% annual interest

- After 1 year: $A = 5000(1.06)^1 = €5,300$
- After 10 years: $A = 5000(1.06)^{10} = €8,954.24$

Scientific Notation in Business

Data Analysis Example:

A tech company processes:

- Daily transactions: 2.4×10^8
- Average transaction value: 3.5×10^1 euros
- Server cost per transaction: 2×10^{-4} euros

Calculate daily revenue and costs:

- Revenue = $(2.4 \times 10^8) \times (3.5 \times 10^1)$ = 8.4×10^9 euros
- Costs = $(2.4 \times 10^8) \times (2 \times 10^{-4})$ = 4.8×10^4 euros
- Profit = €8.4 billion - €48,000 ≈ €8.4 billion

Practice Integration

Individual Exercise 02

Apply your new skills individually

1. Simplify: $\frac{(3x^2)^3 \cdot x^{-5}}{9x^2}$

2. Factor: $4x^2 - 36$
3. Solve: $|3x - 9| = 6$
4. Express in scientific notation: The distance from Earth to Moon is 384,400 km
5. Evaluate: $2 + 3 \times 2^3 - 16 \div 4$

Pair Exercise 02

Business application problem

A manufacturing company has:

- Quality standard: Product weight must satisfy $|w - 100| \leq 2$ grams
- Daily production: 3.2×10^3 unit

a) What is the acceptable weight range?
b) Express monthly production in scientific notation (25 working days)

Wrap-up

Key Takeaways

- PEMDAS ensures consistent calculation order
- Exponent laws are the foundation for all algebra
- Scientific notation handles extreme values efficiently
- Absolute value measures distance and defines tolerances
- Basic factorization reveals structure in expressions

These are the foundation for all advanced mathematics!

For Next Time

Homework: Complete Tasks 01-03

Preview of Session 01-04:

- Advanced factorization techniques
- Roots and radicals
- Complex algebraic manipulation

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> ! Important
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> Master these fundamentals - they're the building blocks for everything else!

Questions & Discussion

Open Floor

Your questions and insights are welcome!

- Any clarifications needed on today's material?

- Connections to other courses?
- Real-world applications you're curious about?

See you next session!

Keep practicing - algebra gets easier with repetition!