Course Cheatsheet

Section 08: Financial Mathematics

Geometric Sequences and Series

Geometric Sequence

A sequence where each term is multiplied by a constant ratio r:

$$a_1, a_1 \cdot r, a_1 \cdot r^2, a_1 \cdot r^3, \ldots$$

Key Formulas

Formula	Expression
nth term	$a_n = a_1 \cdot r^{n-1}$
Recursive	$a_n = a_{n-1} \cdot r$
Common ratio	$r = \frac{a_{n+1}}{a_n}$

Sum of a Geometric Series

Finite sum (n terms):

$$S_n = a_1 \cdot \frac{1 - r^n}{1 - r} = a_1 \cdot \frac{r^n - 1}{r - 1} \quad (r \neq 1)$$

Infinite sum ($|r| < 1$):

$$S_\infty = \frac{a_1}{1 - r}$$

> **i Behavior of Geometric Sequences**
>
> - $|r| > 1$: Sequence grows without bound (or alternates with growth if $r < -1$)
> - $|r| < 1$: Sequence decays toward zero (infinite sum converges)
> - $|r| = 1$: Constant sequence ($r = 1$) or alternating ($r = -1$)
> - $r < 0$: Signs alternate between positive and negative

Arithmetic vs. Geometric Sequences

Property	Arithmetic	Geometric
Pattern	Add constant d	Multiply by constant r
nth term	$a_n = a_1 + (n-1)d$	$a_n = a_1 \cdot r^{n-1}$

Property	Arithmetic	Geometric
Finite sum	$S_n = \frac{n}{2}(a_1 + a_n)$	$S_n = a_1 \cdot \frac{r^n - 1}{r - 1}$
Growth type	Linear	Exponential
Financial use	Simple interest, depreciation	Compound interest, annuities

> 💡 Recognizing the Sequence Type
>
> If the difference between consecutive terms is constant, it is arithmetic. If the ratio between consecutive terms is constant, it is geometric. Compound interest creates geometric sequences because each period multiplies by the same growth factor $(1 + r)$.

Worked Examples for Geometric Sequences

Growth example: A company's revenue doubles every 3 years, starting at 500,000.

- $a_1 = 500,000$, $r = 2$
- After 9 years (3 doublings): $a_4 = 500,000 \cdot 2^3 = 4,000,000$

Decay example: A car loses 15% of its value each year, starting at 30,000.

- $a_1 = 30,000$, $r = 0.85$
- After 5 years: $a_6 = 30,000 \cdot 0.85^5 \approx 13,311$

Series example: Total revenue over 4 years if year 1 is 100 and revenue grows by 10% per year:

- $S_4 = 100 \cdot \frac{1.10^4 - 1}{1.10 - 1} = 100 \cdot \frac{0.4641}{0.10} = 464.10$

Compound Interest

Simple vs. Compound Interest

Type	Formula	Growth
Simple Interest	$A = P(1 + rt)$	Linear
Compound Interest	$A = P(1 + r)^t$	Exponential

> ❗ Why Compound Interest Matters
>
> In compound interest, you earn interest on interest. Over long time horizons, this exponential growth significantly outpaces simple interest. The earlier you invest, the more compounding works in your favor.

General Compound Interest Formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where:

- A = Future value (final amount)
- P = Principal (initial investment)
- r = Annual interest rate (as decimal)
- n = Number of compounding periods per year
- t = Time in years

Compounding Frequency	n
Annual	1
Semi-annual	2
Quarterly	4
Monthly	12
Daily	365

Continuous Compounding

$$A = Pe^{rt}$$

Derivation: $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$

> ⚠️ **Rate and Period Must Always Match**
>
> When payments are monthly, you must use the monthly rate ($r/12$) and the total number of months ($12t$). Never mix an annual rate with a monthly period count – this is one of the most frequent exam errors in financial mathematics.

Effective Annual Rate (EAR)

Purpose

The EAR converts any compounding frequency to an equivalent annual rate, enabling fair comparison between different offers.

Formulas

For periodic compounding:

$$r_{\text{eff}} = \left(1 + \frac{r}{n}\right)^n - 1$$

For continuous compounding:

$$r_{\text{eff}} = e^r - 1$$

Worked Example

Bank A: 6% compounded monthly $\rightarrow r_{\text{eff}} = (1 + 0.06/12)^{12} - 1 = (1.005)^{12} - 1 = 0.0617 = 6.17\%$

Bank B: 6.1% compounded annually $\rightarrow r_{\text{eff}} = 6.10\%$

Bank A is slightly better (6.17% > 6.10%) despite the lower nominal rate.

> 💡 Use EAR for Comparison
>
> Whenever you need to compare two financial products with different compounding frequencies, compute the EAR for each. The higher EAR is better for savings; the lower EAR is better for loans.

Present Value

Formula

$$PV = \frac{FV}{(1+r)^t} = FV \cdot (1+r)^{-t}$$

With multiple compounding:

$$PV = \frac{FV}{\left(1 + \frac{r}{n}\right)^{nt}}$$

Interpretation

Present value answers: "How much must I invest today to have a certain amount in the future?" It discounts a future value back to the present using the interest rate as a discount factor.

Fisher Equation (Real vs. Nominal Returns)

$$r_{\text{real}} \approx r_{\text{nominal}} - r_{\text{inflation}}$$

Exact formula:

$$1 + r_{\text{real}} = \frac{1 + r_{\text{nominal}}}{1 + r_{\text{inflation}}}$$

Rule of 72

Quick Doubling Time Estimate

$$\text{Doubling time} \approx \frac{72}{r\%}$$

where r is the interest rate as a percentage (not a decimal).

Interest Rate	Approximate Doubling Time
3%	24 years
4%	18 years
6%	12 years
8%	9 years
12%	6 years

> **i Why 72?**
>
> For doubling: $2 = (1+r)^t$, so $t = \frac{\ln 2}{\ln(1+r)}$. For small r: $\ln(1+r) \approx r$, giving $t \approx \frac{0.693}{r} \approx \frac{72}{100r}$. The approximation works best for rates between 2% and 15%.

Annuities

Types of Annuities

Type	Payments Made	German Term
Ordinary Annuity	End of each period	Nachschussig
Annuity Due	Beginning of each period	Vorschussig

Future Value of Annuities

Ordinary Annuity:

$$FV = PMT \cdot \frac{(1+r)^n - 1}{r}$$

Annuity Due:

$$FV_{\text{due}} = PMT \cdot \frac{(1+r)^n - 1}{r} \cdot (1+r)$$

Present Value of Annuities

Ordinary Annuity:

$$PV = PMT \cdot \frac{1 - (1+r)^{-n}}{r}$$

Annuity Due:

$$PV_{\text{due}} = PMT \cdot \frac{1 - (1+r)^{-n}}{r} \cdot (1+r)$$

Solving for Payment Amount

Given a savings goal (FV):

$$PMT = \frac{FV \cdot r}{(1+r)^n - 1}$$

Given a loan amount (PV):

$$PMT = PV \cdot \frac{r}{1 - (1+r)^{-n}}$$

Solving for Number of Payments

$$n = -\frac{\ln\left(1 - \frac{PV \cdot r}{PMT}\right)}{\ln(1+r)}$$

> 💡 Ordinary vs. Annuity Due
>
> Most loans use ordinary annuities (payment at end of period). Rent and leases are typically annuities due (payment at beginning). The conversion is simple: Annuity Due = Ordinary Annuity $\times$ $(1 + r)$ for both FV and PV.

> 💡 Choosing Between FV and PV Formulas
>
> - Use FV when you are building up to a target amount (savings, retirement fund)
> - Use PV when you are paying down a debt or valuing a stream of future payments (loans, pensions, lease comparisons)
> - If you know the future goal and need the payment: rearrange FV formula
> - If you know the loan amount and need the payment: rearrange PV formula

Loan Amortization

Monthly Payment Formula

$$PMT = PV \cdot \frac{r}{1 - (1+r)^{-n}}$$

where PV is the loan principal, r is the interest rate per period, and n is the total number of payments.

Building an Amortization Schedule

For each payment k:

1. Interest portion: $\text{Interest}_k = \text{Balance}_{k-1} \times r$
2. Principal portion: $\text{Principal}_k = PMT - \text{Interest}_k$
3. New balance: $\text{Balance}_k = \text{Balance}_{k-1} - \text{Principal}_k$

Worked Example

Loan: 10,000 at 6% annual for 2 years, monthly payments.

$r = 0.06/12 = 0.005, n = 24$

$PMT = 10,000 \times \frac{0.005}{1-(1.005)^{-24}} = 443.21$

Payment	PMT	Interest	Principal	Balance
0	–	–	–	10,000.00
1	443.21	50.00	393.21	9,606.79
2	443.21	48.03	395.18	9,211.61
3	443.21	46.06	397.15	8,814.46
...	...	...	...	...
24	443.21	2.20	441.01	0.00

Outstanding Balance After k Payments

$$B_k = PV \cdot \frac{(1+r)^n - (1+r)^k}{(1+r)^n - 1}$$

Or equivalently:

$$B_k = PMT \cdot \frac{1 - (1+r)^{-(n-k)}}{r}$$

> **! Key Amortization Insight**
>
> Early payments are mostly interest; later payments are mostly principal. This is because the interest portion depends on the outstanding balance, which is largest at the start. As the balance decreases, more of each payment goes toward reducing the principal.

Cost Analysis and Pricing

Cost Function Components

$$K(x) = K_f + K_v(x)$$

where:

- K_f = Fixed costs (rent, insurance, salaries – do not change with quantity)
- $K_v(x)$ = Variable costs (materials, energy, packaging – depend on quantity x)

Common Cost Function Forms

Type	Example	Behavior
Linear	$K(x) = 500 + 3x$	Constant marginal cost
Quadratic	$K(x) = 400 + 10x + 0.02x^2$	Increasing marginal cost
Cubic	$K(x) = 1000 + 50x - 0.5x^2 + 0.01x^3$	First decreasing, then increasing marginal cost

Average Cost Functions

Function	Formula
Average Total Cost	$\bar{K}(x) = \frac{K(x)}{x} = \frac{K_f}{x} + \frac{K_v(x)}{x}$
Variable Cost per Unit	$k_v(x) = \frac{K(x) - K_f}{x} = \frac{K_v(x)}{x}$
Average Fixed Cost	$\frac{K_f}{x}$ (always decreasing)

Pricing Limits

Short-Term Lower Limit Price (Kurzfristige Preisuntergrenze)

$$p_{\text{min, short}} = \min\{k_v(x)\}$$

Finding the minimum:

1. Calculate $k_v(x) = \frac{K(x) - K_f}{x}$
2. Compute $k_{v'}(x)$ and set it equal to zero
3. Solve for x
4. Substitute back to get the minimum value of k_v

> **i Short-Term Interpretation**
>
> At the short-term lower limit price, the company covers its variable costs but not its fixed costs. This is acceptable temporarily when fixed costs are already committed ("sunk"). Producing at this price is better than shutting down completely, since at least variable costs are covered.

Long-Term Lower Limit Price (Langfristige Preisuntergrenze)

$$p_{\text{min, long}} = \min\{\bar{K}(x)\}$$

Finding the minimum:

1. Calculate $\bar{K}(x) = \frac{K(x)}{x}$
2. Compute $\bar{K}'(x)$ and set it equal to zero
3. Solve for x (may require numerical methods for cubic cost functions)
4. Substitute back to get the minimum value of $\bar{K}$

> **i** Long-Term Interpretation
>
> At the long-term lower limit price, the company covers all costs – both fixed and variable. This is the minimum price required for sustainable, long-term production. If the market price falls below this level permanently, the company should exit the market.

Pricing Decision Framework

Price Level	Decision	Reasoning
$p < k_v^{\min}$	Stop production	Losing money on every unit produced
$k_v^{\min} \leq p < \bar{K}^{\min}$	Continue short-term only	Covering variable costs, contributing to fixed costs
$p \geq \bar{K}^{\min}$	Sustainable production	All costs are covered

> **!** Short-Term Limit Is Always Below Long-Term Limit
>
> Since $\bar{K}(x) = k_v(x) + \frac{K_f}{x}$ and $\frac{K_f}{x} > 0$, the average total cost is always above the variable cost per unit. Therefore, $p_{\min,\text{ short}} < p_{\min,\text{ long}}$ always holds.

Contribution Margin (Deckungsbeitrag)

$$\text{Contribution per unit} = p - k_v(x)$$

Interpretation: How much each sold unit "contributes" toward covering fixed costs. If positive, the unit helps reduce the loss from fixed costs; if negative, stop producing immediately.

Total contribution margin: $\text{Total CM} = x \cdot (p - k_v(x))$

Break-Even Analysis

The break-even point is where total revenue equals total cost:

$$p \cdot x = K(x)$$

For a linear cost function $K(x) = K_f + k_v \cdot x$:

$$x_{\text{BE}} = \frac{K_f}{p - k_v}$$

This is equivalent to: fixed costs divided by contribution margin per unit.

> 💡 Connection Between Break-Even and Contribution Margin
>
> The break-even quantity tells you how many units you need to sell to cover all fixed costs. Each unit sold contributes $(p - k_v)$ toward fixed costs. So the number of units needed is simply $\frac{K_f}{p - k_v}$.

Quick Reference: Formula Collection

Compound Interest and Present Value

Concept	Formula
Compound Interest	$A = P(1 + r)^t$
Compound (n periods)	$A = P(1 + r/n)^{nt}$
Continuous	$A = Pe^{rt}$
Present Value	$PV = FV/(1 + r)^t$
Effective Rate	$r_{\text{eff}} = (1 + r/n)^n - 1$
Rule of 72	$t_{\text{double}} \approx 72/r\%$

Annuities

Concept	Formula
FV (Ordinary)	$FV = PMT \cdot \frac{(1+r)^n - 1}{r}$
PV (Ordinary)	$PV = PMT \cdot \frac{1-(1+r)^{-n}}{r}$
Payment (from FV)	$PMT = \frac{FV \cdot r}{(1+r)^n - 1}$
Payment (from PV)	$PMT = PV \cdot \frac{r}{1-(1+r)^{-n}}$
Annuity Due	Multiply ordinary by $(1 + r)$
Outstanding Balance	$B_k = PMT \cdot \frac{1-(1+r)^{-(n-k)}}{r}$

Cost Analysis

Concept	Formula
Total Cost	$K(x) = K_f + K_v(x)$
Variable Cost per Unit	$k_v(x) = \frac{K(x) - K_f}{x}$
Average Total Cost	$\bar{K}(x) = \frac{K(x)}{x}$
Short-term Limit	$\min\{k_v(x)\}$
Long-term Limit	$\min\{\bar{K}(x)\}$
Contribution Margin	$p - k_v(x)$

Concept	Formula
Break-Even (linear)	$x_{\text{BE}} = \frac{K_f}{p-k_v}$

Geometric Sequences

Concept	Formula
nth term	$a_n = a_1 \cdot r^{n-1}$
Finite sum	$S_n = a_1 \cdot \frac{r^n - 1}{r - 1}$
Infinite sum ($|r| < 1$)	$S_\infty = \frac{a_1}{1-r}$

Problem-Solving Strategies

For Compound Interest Problems

1. Identify the given quantities (P, r, n, t) and what you need to find
2. Convert rates to match the compounding period (annual rate / 12 for monthly)
3. Calculate total number of periods ($n \times t$)
4. Apply the appropriate formula
5. Check reasonableness using the Rule of 72

For Annuity Problems

1. Identify the type: ordinary annuity or annuity due
2. Determine what is given and what you need (FV, PV, PMT, or n)
3. Match rate and period (monthly rate for monthly payments)
4. Choose the correct formula (FV for savings goals, PV for loan payoffs)
5. Double-check the sign of the exponent ($-n$ for PV formulas)

For Cost Analysis Problems

1. Identify fixed costs K_f by setting $x = 0$ or reading the constant term
2. Write the variable cost function $K_v(x) = K(x) - K_f$
3. Calculate $k_v(x)$ and $\bar{K}(x)$
4. Find minima using calculus (first derivative = 0, confirm with second derivative)
5. Interpret the results in the context of the pricing decision framework

For Amortization Schedules

1. Calculate the monthly payment using the PV loan formula
2. For each row: compute interest = balance $\times$ rate, principal = PMT $-$ interest
3. Update the balance: new balance = old balance $-$ principal
4. Verify the final balance is (approximately) zero

⚠️ Common Mistakes to Avoid

- Rate/period mismatch: Using an annual rate with monthly periods (or vice versa). Always convert: monthly rate = annual rate / 12, total periods = years $\times$ 12.
- Ordinary vs. annuity due confusion: Most loans are ordinary annuities; rent/leases are annuities due. If unsure, read whether payments are at the "beginning" or "end" of each period.
- Confusing short-term and long-term pricing limits: Short-term = min of $k_v(x)$ (variable cost per unit). Long-term = min of $\bar{K}(x)$ (average total cost). The German terms help: "kurzfristig" = short-term, "langfristig" = long-term.
- Forgetting fixed costs in average cost: $\bar{K}(x) = k_v(x) + \frac{K_f}{x}$. The fixed cost per unit $\frac{K_f}{x}$ decreases as x grows but never reaches zero.
- Not matching compounding periods with payment periods: If payments are monthly, you need the monthly rate and monthly period count everywhere in the formula.
- Arithmetic errors in amortization: Always verify that your schedule's final balance is zero (or very close). If not, recheck the payment calculation.