# Course Cheatsheet

## Section 06: Integral Calculus

## Antiderivatives

### Definition

Antiderivative: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Indefinite Integral Notation:

$$\int f(x)\,dx = F(x) + C$$

where $C$ is the constant of integration.

> ! Always Include +C
>
> For indefinite integrals, always add the constant of integration!

### Basic Antiderivative Rules

| Function | Antiderivative |
|---|---|
| $x^n\ (n \neq -1)$ | $\frac{x^{n+1}}{n+1} + C$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert + C$ |
| $e^x$ | $e^x + C$ |
| $e^{ax}$ | $\frac{1}{a}e^{ax} + C$ |
| $a^x$ | $\frac{a^x}{\ln a} + C$ |
| $\sin x$ | $-\cos x + C$ |
| $\cos x$ | $\sin x + C$ |

### Integration Rules

| Rule | Formula |
|---|---|
| Constant Multiple | $\int k \cdot f(x)\,dx = k \int f(x)\,dx$ |
| Sum/Difference | $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ |

Memory Aid

"Integration undoes differentiation." Ask: "What function, when differentiated, gives me this?"

# Definite Integrals

## Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

Notation: $[F(x)]_a^b = F(b) - F(a)$

Evaluation Process

Always write out the bracket notation: $[F(x)]_a^b = F(b) - F(a)$. This helps avoid sign errors when substituting limits.

## Properties of Definite Integrals

| Property | Formula |
|---|---|
| Constant Multiple | $\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$ |
| Sum | $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ |
| Reversal | $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ |
| Additivity | $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ |
| Zero Width | $\int_a^a f(x)\,dx = 0$ |

## Net vs. Total Area

- Net area (signed): $\int_a^b f(x)\,dx$ (areas below x-axis are negative)
- Total area (unsigned): $\int_a^b |f(x)|\,dx$

# Area Problems

## Area Under a Curve

For $f(x) \geq 0$ on $[a, b]$:

$$\text{Area} = \int_a^b f(x)\,dx$$

For functions that cross the x-axis, split at zeros:

$$\text{Total Area} = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

(where $f(c) = 0$ and $f(x) < 0$ on $(c, b)$)

### Area Between Curves

For $f(x) \geq g(x)$ on $[a, b]$:

$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx$$

> ! Finding the Upper Function
>
> Always determine which function is "on top" by testing a point in the interval: - If $f(c) > g(c)$, then $f$ is the upper function - Use: Upper minus Lower in the integrand

### Steps for Area Between Curves

1. Find intersection points: Solve $f(x) = g(x)$
2. Determine which function is on top in each region
3. Set up integral(s): $\int_a^b [\text{top} - \text{bottom}]\,dx$
4. Evaluate using the Fundamental Theorem

## Economic Applications

### Consumer Surplus

$$CS = \int_0^{q^*} [D(q) - p^*]\,dq$$

where: - $D(q)$ = demand function - $q^*$ = equilibrium quantity - $p^*$ = equilibrium price

Interpretation: Total benefit consumers receive from paying less than their maximum willingness to pay.

> Geometric Interpretation
>
> Consumer surplus is the area between the demand curve and the horizontal price line. Sketch the region to visualize what you're calculating!

### Producer Surplus

$$PS = \int_0^{q^*} [p^* - S(q)]\,dq$$

where: - $S(q)$ = supply function - $q^*$ = equilibrium quantity - $p^*$ = equilibrium price

Interpretation: Total benefit producers receive from selling at a price higher than their minimum acceptable price.

### Finding Equilibrium

At equilibrium: $D(q) = S(q)$

Solve to find $q^*$, then $p^* = D(q^*) = S(q^*)$

### Total Cost from Marginal Cost

$$C(x) = \int MC(x)\,dx = \int C'(x)\,dx$$

Use initial condition $C(0) = \text{Fixed Costs}$ to find $C$.

### Revenue from Marginal Revenue

$$R(x) = \int MR(x)\,dx = \int R'(x)\,dx$$

Typically $R(0) = 0$ (no revenue with no sales).

## Average Value of a Function

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Interpretation: The constant height of a rectangle with the same area as the region under the curve.

> Visual Understanding
>
> The average value is like “flattening” the curve into a horizontal line with equal area.

## Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

### LIATE Rule for Choosing u

Choose $u$ from this list (first available):

1. Logarithmic functions ($\ln x$)
2. Inverse trigonometric functions
3. Algebraic functions ($x^n$, polynomials)
4. Trigonometric functions
5. Exponential functions ($e^x$)

Common Integration by Parts Formulas

| Integral | Result |
|---|---|
| $\int xe^x\,dx$ | $e^x(x-1)+C$ |
| $\int x^n \ln x\,dx$ | $\frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right)+C$ |
| $\int \ln x\,dx$ | $x(\ln x - 1)+C$ |

# Initial Value Problems

Given: $f'(x)=g(x)$ and $f(a)=b$

Steps:

1. Find general antiderivative: $f(x)=\int g(x)\,dx = G(x)+C$
2. Apply initial condition: $f(a)=G(a)+C=b$
3. Solve for $C$
4. Write specific solution

# Quick Reference: Common Integrals

| Function | Antiderivative |
|---|---|
| $k$ (constant) | $kx+C$ |
| $x^n$ | $\frac{x^{n+1}}{n+1}+C$ ($n \neq -1$) |
| $\frac{1}{x}$ | $\ln\lvert x\rvert + C$ |
| $e^x$ | $e^x+C$ |
| $e^{ax}$ | $\frac{e^{ax}}{a}+C$ |
| $\sqrt{x}$ | $\frac{2x^{3/2}}{3}+C$ |
| $\frac{1}{x^2}$ | $-\frac{1}{x}+C$ |

# Problem-Solving Strategies

## For Indefinite Integrals

1. Simplify first if possible (expand, factor out constants)
2. Rewrite radicals and fractions as powers
3. Apply rules term by term
4. Verify by differentiating the result

## For Definite Integrals

1. Find the antiderivative (without +C)
2. Evaluate at upper limit
3. Evaluate at lower limit
4. Subtract: $F(b)-F(a)$

### For Area Problems

1. Sketch the region (if possible)
2. Find boundaries and intersection points
3. Identify upper and lower functions
4. Set up the integral(s)
5. Calculate and interpret

### Common Mistakes to Avoid

- Forgetting the $+C$ for indefinite integrals
- Wrong sign when function is below x-axis
- Mixing up upper and lower functions in area problems
- Forgetting to apply both limits in definite integrals
- Not simplifying before integrating