Course Cheatsheet

Section 05: Differential Calculus

Limits

Limit Notation

Definition: $\lim_{x \to a} f(x) = L$ means $f(x)$ approaches L as x approaches a

Key Properties:

- $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- $\lim_{x \to a}[c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$
- $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ (if denominator $\neq 0$)

One-Sided Limits

Notation	Meaning
$\lim_{x \to a^-} f(x)$	Limit from the left (values less than a)
$\lim_{x \to a^+} f(x)$	Limit from the right (values greater than a)

Limit exists if and only if: $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

Limits at Infinity

For rational functions $\frac{P(x)}{Q(x)}$:

Degree Comparison	Result
deg(P) < deg(Q)	$\lim_{x \to \infty} = 0$
deg(P) = deg(Q)	$\lim_{x \to \infty} = \frac{a_n}{b_n}$ (ratio of leading coefficients)
deg(P) > deg(Q)	$\lim_{x \to \infty} = \pm\infty$

Indeterminate Forms

Forms that require further analysis:

- $\frac{0}{0}$ — Factor and cancel, or use L'Hôpital's rule
- $\frac{\infty}{\infty}$ — Divide by highest power, or use L'Hôpital's rule

Not indeterminate:

- $\frac{k}{0}$ where $k \neq 0 \to \pm\infty$ (vertical asymptote)
- $\frac{0}{k}$ where $k \neq 0 \to 0$

Continuity

Three Conditions for Continuity at $x = a$

1. $f(a)$ exists (function is defined at a)
2. $\lim_{x\to a} f(x)$ exists (limit exists)
3. $\lim_{x\to a} f(x) = f(a)$ (limit equals function value)

Types of Discontinuities

Type	Description	Example
Removable	Limit exists but $\neq f(a)$ or $f(a)$ undefined	Hole in graph
Jump	Left and right limits exist but are different	Step function
Infinite	Function approaches $\pm\infty$	Vertical asymptote

! Visual Continuity Test

If you can draw the graph without lifting your pen, the function is continuous at that point.

Derivatives

Definition

Formal Definition:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Alternative Form:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

Notation

Notation	Read as
$f'(x)$	"f prime of x"
$\frac{dy}{dx}$	"dy dx" (Leibniz notation)
$\frac{d}{dx}[f(x)]$	"derivative of f with respect to x"

Notation	Read as
$\frac{df}{dx}\Big\|_{x=a}$	Derivative evaluated at $x = a$

Geometric Interpretation

- $f'(a)$ = slope of tangent line to $f(x)$ at $x = a$
- Tangent line equation: $y - f(a) = f'(a)(x - a)$

Economic Interpretation

Function	Derivative	Interpretation
$C(x)$ = Total Cost	$C'(x)$ = Marginal Cost	Cost of producing one more unit
$R(x)$ = Revenue	$R'(x)$ = Marginal Revenue	Revenue from selling one more unit
$P(x)$ = Profit	$P'(x)$ = Marginal Profit	Profit from one more unit

Differentiation Rules

Basic Rules

Rule	Formula
Constant	$\frac{d}{dx}[c] = 0$
Power	$\frac{d}{dx}[x^n] = nx^{n-1}$
Constant Multiple	$\frac{d}{dx}[cf(x)] = c \cdot f'(x)$
Sum/Difference	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Product and Quotient Rules

Product Rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

> Memory Aid for Quotient Rule
>
> "Low d-high minus high d-low, over the square of what's below"

Chain Rule

For composite functions $f(g(x))$:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Leibniz notation: If $y = f(u)$ and $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Common Patterns:

Function	Derivative
$(ax+b)^n$	$n(ax+b)^{n-1} \cdot a$
$\sqrt{f(x)}$	$\frac{f'(x)}{2\sqrt{f(x)}}$
$\frac{1}{f(x)}$	$\frac{-f'(x)}{[f(x)]^2}$

> ! Most Common Mistake
>
> Forgetting to multiply by the inner derivative! Always ask: "What's inside?"

Implicit Differentiation

When to Use

- Variables are intertwined (can't easily solve for y)
- Equations like $x^2 + y^2 = 25$, $xy = k$, or $L^{0.6}K^{0.4} = 100$

Technique

1. Differentiate both sides with respect to x
2. Apply chain rule to terms with y: $\frac{d}{dx}[y^n] = ny^{n-1} \cdot \frac{dy}{dx}$
3. Collect all $\frac{dy}{dx}$ terms on one side
4. Solve for $\frac{dy}{dx}$

Example

For $xy = 5000$:

$$\frac{d}{dx}[xy] = \frac{d}{dx}[5000]$$

$$y + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Related Rates

Process

1. Identify all variables and their relationships
2. Write an equation relating the quantities
3. Differentiate both sides with respect to time t
4. Substitute known values
5. Solve for the unknown rate

Common Formulas

For quantity Q depending on variable x:

$$\frac{dQ}{dt} = \frac{dQ}{dx} \cdot \frac{dx}{dt}$$

Example: If $R = 50\sqrt{C}$ and customers grow at rate $\frac{dC}{dt}$:

$$\frac{dR}{dt} = \frac{25}{\sqrt{C}} \cdot \frac{dC}{dt}$$

Graphical Calculus

From $f(x)$ to $f'(x)$

Feature of $f(x)$	Feature of $f'(x)$
Increasing	$f'(x) > 0$ (positive)
Decreasing	$f'(x) < 0$ (negative)
Local maximum	$f'(x) = 0$ (crosses from + to −)
Local minimum	$f'(x) = 0$ (crosses from − to +)
Steep slope	Large $\lvert f'(x) \rvert$
Horizontal tangent	$f'(x) = 0$

From $f'(x)$ to $f(x)$

Feature of $f'(x)$	Feature of $f(x)$
$f'(x) > 0$	f is increasing
$f'(x) < 0$	f is decreasing
$f'(x) = 0$	f has horizontal tangent (possible extremum)
f' increasing	f is concave up
f' decreasing	f is concave down

Critical Points

Definition: $x = c$ is a critical point if $f'(c) = 0$ or $f'(c)$ is undefined

First Derivative Test:

Sign change of $f'(x)$	Type of critical point
1. to –	Local maximum
– to +	Local minimum
No sign change	Neither (inflection point)

Second Derivative

Concavity:

- $f''(x) > 0$: Concave up (smile) ∪
- $f''(x) < 0$: Concave down (frown) ∩

Inflection Point: Where $f''(x) = 0$ and concavity changes

Second Derivative Test:

- If $f'(c) = 0$ and $f''(c) > 0$: Local minimum
- If $f'(c) = 0$ and $f''(c) < 0$: Local maximum
- If $f'(c) = 0$ and $f''(c) = 0$: Test is inconclusive

Business Applications

Marginal Analysis

Key Relationships:

- Profit is maximized when $P'(x) = 0$, equivalently when $R'(x) = C'(x)$
- Produce more if $R'(x) > C'(x)$ (marginal revenue exceeds marginal cost)
- Produce less if $R'(x) < C'(x)$

Optimization Strategy

1. Find the function to optimize (profit, cost, revenue)
2. Take the derivative and set equal to zero
3. Solve for critical points
4. Test using second derivative or endpoints
5. Verify the answer makes business sense

Average Cost

Average Cost Function: $AC(x) = \frac{C(x)}{x}$

Minimum average cost occurs when $AC'(x) = 0$, which happens when $AC(x) = C'(x)$ (average cost equals marginal cost)

Quick Reference: Derivative Rules

Function	Derivative
c (constant)	0
x^n	nx^{n-1}
$\sqrt{x} = x^{1/2}$	$\frac{1}{2\sqrt{x}}$
$\frac{1}{x} = x^{-1}$	$-\frac{1}{x^2}$
$\frac{1}{x^n} = x^{-n}$	$-\frac{n}{x^{n+1}}$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$

Problem-Solving Strategies

Finding Derivatives

1. Simplify first if possible (rewrite radicals as powers)
2. Identify which rule(s) apply (power, product, quotient, chain)
3. Work systematically — don't skip steps
4. Simplify the result

Optimization Problems

1. Draw a picture if applicable
2. Define variables and write the objective function
3. Express in terms of one variable using constraints
4. Differentiate and find critical points
5. Check endpoints if domain is restricted
6. Interpret the result in context

Common Mistakes to Avoid

- Forgetting the chain rule inner derivative
- Using power rule on products: $(fg)' \neq f' \cdot g'$
- Forgetting that $\frac{dy}{dx}$ appears when differentiating y terms implicitly
- Not checking if critical points are maxima or minima
- Forgetting units in applied problems