Course Cheatsheet

Section 04: Advanced Functions

Polynomial Functions

General Form

Polynomial of degree n: $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$

Where:

- $a_n \neq 0$ (leading coefficient)
- n is the degree (highest power)
- Domain: All real numbers

Polynomial Classification by Degree

Degree	Name	General Form	Example
0	Constant	$f(x) = a$	$f(x) = 5$
1	Linear	$f(x) = ax + b$	$f(x) = 2x + 3$
2	Quadratic	$f(x) = ax^2 + bx + c$	$f(x) = x^2 - 4$
3	Cubic	$f(x) = ax^3 + bx^2 + cx + d$	$f(x) = 2x^3 - x$
4	Quartic	$f(x) = ax^4 + ...$	$f(x) = x^4 - 5x^2 + 4$

End Behavior Rules

Determined by degree and leading coefficient:

Degree	Leading Coefficient	Left End	Right End	Shape
Even	Positive ($a > 0$)	↑	↑	Both ends up
Even	Negative ($a < 0$)	↓	↓	Both ends down
Odd	Positive ($a > 0$)	↓	↑	Left down, right up
Odd	Negative ($a < 0$)	↑	↓	Left up, right down

Key Properties

Number of Zeros (x-intercepts):

- Maximum possible: equal to degree
- Can be less if roots are repeated or complex

Number of Turning Points:

- Maximum: degree - 1
- A turning point is where function changes from increasing to decreasing (or vice versa)

Multiplicity of Zeros:

- Odd multiplicity: Graph crosses x-axis
- Even multiplicity: Graph touches x-axis but doesn't cross

Power Functions & Roots

Power Functions

General form: $f(x) = kx^p$

Where k is coefficient and p is the power

Common Power Functions:

Function	Domain	Range	Key Features
$f(x) = x^2$	All real	$[0, \infty)$	Parabola, even function
$f(x) = x^3$	All real	All real	Cubic, odd function
$f(x) = x^{1/2} = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$	Square root
$f(x) = x^{-1} = \frac{1}{x}$	$x \neq 0$	$y \neq 0$	Reciprocal/Hyperbola

Root Functions

Square Root: $f(x) = \sqrt{x}$

- Domain: $x \geq 0$
- Range: $[0, \infty)$
- Always increasing

Cube Root: $f(x) = \sqrt[3]{x}$

- Domain: All real numbers
- Range: All real numbers
- Always increasing
- Passes through origin

General Root: $f(x) = \sqrt[n]{x}$

- Even n: Domain restricted to $x \geq 0$
- Odd n: Domain is all real numbers

Rational Exponents

Conversion Rules:

- $x^{1/n} = \sqrt[n]{x}$
- $x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

- $x^{-n} = \frac{1}{x^n}$

Properties:

- $(x^a)^b = x^{ab}$
- $x^a \cdot x^b = x^{a+b}$
- $\frac{x^a}{x^b} = x^{a-b}$

Exponential Functions

Definition and Properties

Exponential function: $f(x) = a \cdot b^x$ where $b > 0, b \neq 1$

Key Components:

- a: Initial value (when $x = 0$)
- b: Base (growth/decay factor)
- x: Exponent (time, periods)

Growth vs Decay

Type	Base	Behavior	Example
Exponential Growth	$b > 1$	Increases rapidly	$f(x) = 2^x$
Exponential Decay	$0 < b < 1$	Decreases rapidly	$f(x) = (0.5)^x$

Standard Form: $f(t) = P_0 \cdot (1 + r)^t$

- P_0: Initial amount
- r: Growth rate (positive) or decay rate (negative)
- t: Time

The Natural Exponential: e

Euler's number: $e \approx 2.71828$

Continuous Growth: $f(t) = Pe^{rt}$

- Most natural model for continuous processes
- $r > 0$: continuous growth
- $r < 0$: continuous decay

Applications:

- Compound interest: $A = Pe^{rt}$
- Population growth
- Radioactive decay
- Bacterial growth

Exponential Properties

Essential Rules:

- $b^x \cdot b^y = b^{x+y}$
- $\frac{b^x}{b^y} = b^{x-y}$
- $(b^x)^y = b^{xy}$
- $b^0 = 1$
- $b^{-x} = \frac{1}{b^x}$

Doubling and Half-Life

Doubling Time: Time for quantity to double - If $P(t) = P_0 \cdot 2^{t/T_d}$, then T_d is doubling time

Half-Life: Time for quantity to reduce by half - If $P(t) = P_0 \cdot (0.5)^{t/T_h}$, then T_h is half-life

Logarithmic Functions

Definition

Logarithm is the inverse of exponential:

$$\log_b(x) = y \Leftrightarrow b^y = x$$

Key Points:

- $\log_b(b) = 1$
- $\log_b(1) = 0$
- $\log_b(b^x) = x$
- $b^{\log_b(x)} = x$

Common Logarithms

Common Log (base 10): $\log(x)$ or $\log_{10}(x)$ - Used in science, pH scale, Richter scale

Natural Log (base e): $\ln(x)$ or $\log_e(x)$ - Used in calculus, continuous growth - $\ln(e) = 1$

Logarithm Properties

Product Rule: $\log_b(xy) = \log_b(x) + \log_b(y)$

Quotient Rule: $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

Power Rule: $\log_b(x^n) = n \log_b(x)$

Change of Base: $\log_b(x) = \frac{\log_a(x)}{\log_a(b)} = \frac{\ln(x)}{\ln(b)}$

Domain and Range

For $\boldsymbol{f(x) = \log_b(x)}$:

- Domain: $(0, \infty)$ (only positive numbers)
- Range: All real numbers

- Vertical asymptote at $x = 0$
- Passes through $(1, 0)$ and $(b, 1)$

Solving Logarithmic Equations

Strategy 1: Combine logs

$$\log(x) + \log(x - 3) = 1$$

$$\log(x(x - 3)) = 1$$

$$x(x - 3) = 10^1$$

Strategy 2: Convert to exponential

$$\log_2(x) = 5$$

$$x = 2^5 = 32$$

> ! Always Check Domain!
>
> Solutions must satisfy $x > 0$ for all logarithmic arguments

Trigonometric Functions

The Unit Circle

Unit circle definition:

- Circle with radius 1 centered at origin
- Point on circle: $(\cos\theta, \sin\theta)$

Key Angles and Values:

Angle	Degrees	Radians	$\sin$	$\cos$	$\tan$
0°	0°	0	0	1	0
30°	30°	$\pi/6$	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$
45°	45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	60°	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90°	90°	$\pi/2$	1	0	undefined

Degrees and Radians

Conversion:

- $180° = \pi$ radians
- Degrees to radians: multiply by $\frac{\pi}{180}$
- Radians to degrees: multiply by $\frac{180}{\pi}$

Why radians?

- Arc length: $s = r\theta$ (when θ in radians)
- Natural for calculus
- Simplifies many formulas

Basic Trigonometric Functions

Sine Function: $f(x) = \sin(x)$

- Domain: All real numbers
- Range: $[-1, 1]$
- Period: 2π
- Starts at origin

Cosine Function: $f(x) = \cos(x)$

- Domain: All real numbers
- Range: $[-1, 1]$
- Period: 2π
- Starts at maximum (1)

Tangent Function: $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$

- Domain: All real except $x = \frac{\pi}{2} + n\pi$
- Range: All real numbers
- Period: π
- Vertical asymptotes where $\cos(x) = 0$

Key Trigonometric Identities

Reciprocal Identities:

- $\csc(x) = \frac{1}{\sin(x)}$
- $\sec(x) = \frac{1}{\cos(x)}$
- $\cot(x) = \frac{1}{\tan(x)}$

Even/Odd Properties:

- $\cos(-x) = \cos(x)$ (even)
- $\sin(-x) = -\sin(x)$ (odd)

Transformations of Trigonometric Functions

General form: $f(x) = A\sin(B(x - C)) + D$

Parameter	Effect	Name
A	Vertical stretch by $\| A \|$	Amplitude
B	Horizontal compression by $\| B \|$	Frequency factor
C	Horizontal shift right by C	Phase shift

Parameter	Effect	Name
D	Vertical shift up by D	Midline

Period: $\frac{2\pi}{|B|}$ (for sine and cosine)

Example: $f(x) = 3\sin(2x - \pi) + 1$

- Amplitude: 3
- Period: $\frac{2\pi}{2} = \pi$
- Phase shift: $\frac{\pi}{2}$ right
- Midline: $y = 1$

Business Applications

Seasonal Patterns:

- Sales cycles
- Temperature variations
- Demand fluctuations

Example: Seasonal Revenue

$$R(t) = 50 + 20\sin\left(\frac{2\pi}{12}(t - 3)\right)$$

- Average revenue: €50k - Seasonal variation: ±€20k - Period: 12 months - Peak: Month 6 (June)

Function Transformations (Universal)

The Transformation Framework

Standard form: $g(x) = a \cdot f(b(x - h)) + k$

Order of transformations:

1. Horizontal shift: h (inside function)
2. Horizontal stretch/compress: b (inside function)
3. Vertical stretch/compress: a (outside function)
4. Vertical shift: k (outside function)

Universal Transformation Rules

Transformation	Formula	Effect
Shift up	$f(x) + k$	Move up k units
Shift down	$f(x) - k$	Move down k units
Shift right	$f(x - h)$	Move right h units
Shift left	$f(x + h)$	Move left h units

Transformation	Formula	Effect
Vertical stretch	$a \cdot f(x), \| a \| > 1$	Stretch by factor $\| a \|$
Vertical compress	$a \cdot f(x), 0 < \| a \| < 1$	Compress by factor $\| a \|$
Horizontal stretch	$f(bx), 0 < \| b \| < 1$	Stretch by factor $1/ \| b \|$
Horizontal compress	$f(bx), \| b \| > 1$	Compress by factor $1/ \| b \|$
Reflect over x-axis	$-f(x)$	Flip upside down
Reflect over y-axis	$f(-x)$	Flip left-right

! Inside vs Outside

- Inside changes (affect x): Work opposite to intuition
- Outside changes (affect y): Work as expected

The 4-Step Method for Transformations

1. Identify base function
2. Find key points (intercepts, max/min, asymptotes)
3. Track transformations systematically
4. Verify with test point

Rational Functions

Definition and Structure

Rational function: $f(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials

Key components:

- Numerator $P(x)$: Determines zeros (x-intercepts)
- Denominator $Q(x)$: Determines vertical asymptotes
- Degree comparison: Determines horizontal/oblique asymptotes

Domain

Domain: All real numbers except where $Q(x) = 0$

Steps to find domain: 1. Set denominator equal to zero 2. Solve for x 3. Exclude these values from domain

Asymptotes and Holes

Vertical Asymptotes:

- Occur where denominator = 0 (after cancellation)
- Graph approaches $\pm\infty$
- To find: Solve $Q(x) = 0$ after simplifying

Horizontal Asymptotes (comparing degrees):

Condition	Horizontal Asymptote
deg(P) < deg(Q)	$y = 0$
deg(P) = deg(Q)	$y = \frac{a_n}{b_n}$ (ratio of leading coefficients)
deg(P) > deg(Q)	No horizontal asymptote

Oblique (Slant) Asymptotes:

- When deg(P) = deg(Q) + 1
- Find by polynomial long division
- Graph approaches this line as $x \to \pm\infty$

Holes (Removable Discontinuities):

- Occur when factor cancels from numerator and denominator
- To find: Factor completely, cancel common factors
- Point where hole occurs: $(a, f(a))$ where factor is $(x - a)$

Systematic Analysis Process

Always follow this order:

1. Factor completely (both numerator and denominator)
2. Cancel common factors → These become holes
3. Find vertical asymptotes (remaining zeros of denominator)
4. Find horizontal/oblique asymptotes (degree comparison)
5. Find x-intercepts (zeros of simplified numerator)
6. Find y-intercept (evaluate at $x = 0$ if defined)

Example: $f(x) = \frac{x^2-4}{x^2-x-2}$

1. Factor: $f(x) = \frac{(x-2)(x+2)}{(x-2)(x+1)}$
2. Cancel: $f(x) = \frac{x+2}{x+1}$, $x \neq 2$ (hole at $x = 2$)
3. Vertical asymptote: $x = -1$
4. Horizontal asymptote: $y = 1$ (equal degrees, ratio 1/1)
5. x-intercept: $x = -2$
6. y-intercept: $f(0) = 2$
7. Hole at: $(2, \frac{4}{3})$

Business Applications

Average Cost Functions:

$$AC(x) = \frac{C(x)}{x} = \frac{FC + VC \cdot x}{x} = \frac{FC}{x} + VC$$

- Vertical asymptote at $x = 0$
- Horizontal asymptote at $y = VC$ (variable cost per unit)
- As production increases, average cost approaches variable cost

Business Applications Summary

Exponential Growth/Decay Models

Investment Growth:

$$A(t) = P(1 + r)^t \quad \text{or} \quad A(t) = Pe^{rt}$$

Population Models:

- Growth: $P(t) = P_0 e^{kt}$ where $k > 0$
- Decay: $P(t) = P_0 e^{-kt}$ where $k > 0$

Depreciation:

- Declining balance: $V(t) = V_0(1 - r)^t$

Periodic Business Patterns

Revenue Seasonality:

$$R(t) = R_0 + A \sin\left(\frac{2\pi}{12}(t - \phi)\right)$$

Where:

- R_0: Average revenue
- A: Seasonal amplitude
- ϕ: Phase shift (timing of peak)

Optimization with Functions

Revenue Maximization:

- For quadratic revenue: vertex formula
- For rational functions: calculus (Section 05)

Cost Minimization:

- Average cost minimum
- Production efficiency

Problem-Solving Strategies

General Approach

1. Identify function type (polynomial, exponential, rational, etc.)
2. Check domain restrictions (denominators, square roots, logs)
3. Apply appropriate techniques
4. Verify results make practical sense

5. Consider constraints (production limits, budget)

Common Mistakes to Avoid

- Logarithms: Forgetting domain restrictions ($x > 0$)
- Rational functions: Not factoring completely before finding asymptotes
- Transformations: Confusing inside vs outside changes
- Trigonometry: Mixing degrees and radians
- Exponential: Confusing growth rate with growth factor

Quick Reference: When to Use Each Function

Business Situation	Function Type	Example
Constant growth/decay rate	Exponential	$P(t) = P_0 \cdot b^t$
Seasonal patterns	Trigonometric	$S(t) = A\sin(Bt) + C$
Average cost	Rational	$AC(x) = \frac{C(x)}{x}$
Revenue optimization	Quadratic	$R(x) = px = p(a - bp)$
Multi-step processes	Composition	$(f \circ g)(x)$
Reverse relationships	Inverse	$f^{-1}(x)$