# Course Cheatsheet

## Section 03: Functions as Business Models

## What is a Function?

A function is a rule that assigns to each input exactly one output

- Mathematical Definition: For every input $x$, there is exactly one output $f(x)$
- Business Perspective: A model showing cause and effect relationships
- Key Principle: Each input must have exactly one output (no ambiguity)

## Function Notation

| Traditional Equation | Function Notation | Meaning |
|---|---|---|
| $y = 2x + 5$ | $f(x) = 2x + 5$ | Function $f$ maps input $x$ to output $2x + 5$ |
| $y = x^2 - 3$ | $g(x) = x^2 - 3$ | Function $g$ maps input $x$ to output $x^2 - 3$ |
| $y = \frac{100}{x}$ | $h(x) = \frac{100}{x}$ | Function $h$ maps input $x$ to output $\frac{100}{x}$ |

Function Evaluation:

- $f(3)$ means "substitute $x = 3$ into function $f$"
- If $f(x) = 2x + 5$, then $f(3) = 2(3) + 5 = 11$
- If $g(x) = x^2 - 3$, then $g(-2) = (-2)^2 - 3 = 1$

## Domain and Range

### Domain: All Possible Input Values

Mathematical Restrictions:

| Function Type | Restriction | Domain Rule |
|---|---|---|
| Rational: $f(x) = \frac{1}{x-a}$ | Denominator ≠ 0 | $x \neq a$ |
| Square Root: $f(x) = \sqrt{x-a}$ | Argument ≥ 0 | $x \geq a$ |

| Function Type | Restriction | Domain Rule |
|---|---|---|
| Logarithm: $f(x) = \log(x - a)$ | Argument > 0 | $x > a$ |
| Polynomial: $f(x) = ax^n + \dots$ | No restrictions | All real numbers |

Business Restrictions:

- Quantities cannot be negative: $x \geq 0$
- Production capacity limits: $0 \leq x \leq \text{max capacity}$
- Time constraints: $0 \leq t \leq 24$ hours per day

### Range: All Possible Output Values

Common Patterns:

| Function Type | Range | Example |
|---|---|---|
| Linear: $f(x) = ax + b$ | All real numbers | $(-\infty, \infty)$ |
| Quadratic: $f(x) = x^2 + c$ | $[c, \infty)$ | $f(x) = x^2 + 2$ has range $[2, \infty)$ |
| Square Root: $f(x) = \sqrt{x}$ | $[0, \infty)$ | Non-negative outputs |
| Rational: $f(x) = \frac{1}{x}$ | All except 0 | $(-\infty, 0) \cup (0, \infty)$ |

## Four Ways to Represent Functions

### 1. Verbal Description

"Base cost of 100€ plus 3€ for each additional unit"

### 2. Algebraic Formula

$C(x) = 100 + 3x$

### 3. Numerical Table

| $x$ | $C(x)$ |
|---|---|
| 0 | 100 |
| 10 | 130 |
| 20 | 160 |

### 4. Graphical Plot

Visual representation showing the relationship between input and output

## The Vertical Line Test

Rule: A graph represents a function if and only if every vertical line intersects it at most once.

Examples:

- Pass: Lines, parabolas opening up/down, exponential curves
- Fail: Circles, sideways parabolas, vertical lines

Why Important: Ensures each input has exactly one output (function definition)

## Function Evaluation Strategies

### Direct Substitution

If $f(x) = 3x^2 - 5x + 2$, find $f(4)$: - $f(4) = 3(4)^2 - 5(4) + 2 = 3(16) - 20 + 2 = 48 - 20 + 2 = 30$

### Piecewise Functions

When function has different rules for different input ranges:

$$f(x) = \begin{cases} 2x + 1 & \text{if } \; x < 0 \\ x^2 & \text{if } \; x \geq 0 \end{cases}$$

- $f(-3) = 2(-3) + 1 = -5$ (use first rule since $-3 < 0$)
- $f(5) = 5^2 = 25$ (use second rule since $5 \geq 0$)

## Key Vocabulary

- Function: Rule assigning exactly one output to each input
- Domain: Set of all possible input values
- Range: Set of all possible output values
- Independent Variable: Input variable (usually $x$)
- Dependent Variable: Output variable (usually $y$ or $f(x)$)
- Function Notation: $f(x)$ read as "$f$ of $x$"
- Vertical Line Test: Method to determine if graph represents function
- Fixed Costs: Business costs independent of production level
- Variable Costs: Business costs proportional to production level
- Break-Even Point: Where revenue equals cost (profit = 0)

## Linear Functions in Detail

### Forms of Linear Functions

1. Slope-Intercept Form: $y = mx + b$

- m: slope (rate of change/marginal change)
  - Positive: increasing function
  - Negative: decreasing function
  - Zero: horizontal line
- b: y-intercept (starting value/base value)
  - Value when $x = 0$
  - Often represents fixed costs or initial values

2. Point-Slope Form: $y - y_1 = m(x - x_1)$

- Useful when you know one point $(x_1, y_1)$ and the slope $m$
- Best for modeling from observed data

Converting Between Forms:

- From two points $(x_1, y_1)$ and $(x_2, y_2)$:
  - Slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$
  - Use point-slope form, then simplify to slope-intercept

# Supply and Demand Functions

## Demand Functions

Demand shows how quantity purchased depends on price

- General form: $Q_d = a - bp$ (decreasing function)
  - $a$: maximum quantity when price = 0
  - $b$: price sensitivity (how much demand drops per unit price increase)
- Alternative form: $p = c - dQ_d$ (price as function of quantity)

## Supply Functions

Supply shows how quantity produced depends on price

- General form: $Q_s = -c + dp$ (increasing function)
  - Often has positive y-intercept
  - $d$: production response to price changes

## Market Equilibrium

Equilibrium occurs where supply equals demand: $Q_d = Q_s$

- Equilibrium price ($\boldsymbol{p}^*$): Market-clearing price
- Equilibrium quantity ($\boldsymbol{Q}^*$): Amount actually traded
- Graphically: Intersection point of supply and demand curves
- No shortage or surplus at equilibrium

Example:

- Demand: $Q_d = 500 - 50p$
- Supply: $Q_s = -100 + 100p$
- Set equal: $500 - 50p = -100 + 100p$
- Solve: $600 = 150p$, so $p^* = 4$
- Quantity: $Q^* = 500 - 50(4) = 300$ units

## Cost-Volume-Profit

### CVP Framework Components

| Component | Formula | Description |
|---|---|---|
| Fixed Costs (FC) | Constant | Independent of production volume |
| Variable Costs per unit (VC) | Per unit amount | Changes with production |
| Total Costs | $TC = FC + VC \times Q$ | Sum of fixed and variable costs |
| Revenue | $R = P \times Q$ | Price × Quantity |
| Profit | $\Pi = R - TC$ | Revenue minus total costs |
| Contribution Margin | $CM = P - VC$ | Profit per unit before fixed costs |

### CVP Key Calculations

Break-Even Point: $Q_{BE} = \frac{FC}{CM} = \frac{FC}{P - VC}$

Target Profit: $Q_{target} = \frac{FC + \text{Target Profit}}{CM}$

Margin of Safety: Actual sales - Break-even sales

## Linear Modeling from Data

### Steps to Create Linear Models

1. Identify variables (independent vs dependent)
2. Calculate slope between data points
3. Use point-slope form to find equation
4. Interpret parameters in business context

Example: Sales Forecasting

- Data shows consistent increase of 15 units per month
- Starting at 120 units in month 1
- Model: $S(m) = 15m + 105$

## Depreciation Models

### Linear (Straight-Line) Depreciation

Formula: $V(t) = V_0 - dt$

Where:

- $V(t)$: Value at time $t$

- $V_0$: Initial value
- $d$: Depreciation rate per period
- Useful life: $n = \frac{V_0}{d}$ periods

Example: Company Vehicle

- Purchase price: €30,000
- Annual depreciation: €5,000
- Function: $V(t) = 30,000 - 5,000t$
- Fully depreciated after 6 years

# Quadratic Functions

## Standard Form

Formula: $f(x) = ax^2 + bx + c$

Key Components: - a: Direction and width - $a > 0$: Opens upward (U-shape, has minimum) - $a < 0$: Opens downward (∩-shape, has maximum) - $|a|$ larger → Narrower parabola - b: Affects position of vertex - c: y-intercept (value when $x = 0$)

Graph Shape: Parabola (curved, not straight like linear functions)

## The Vertex Formula

Key Formula: $x_v = -\frac{b}{2a}$

For quadratic function $f(x) = ax^2 + bx + c$: - Vertex x-coordinate: $x_v = -\frac{b}{2a}$ - Vertex y-coordinate: $f(x_v) = f\left(-\frac{b}{2a}\right)$ - Vertex represents: - Maximum if $a < 0$ (parabola opens down) - Minimum if $a > 0$ (parabola opens up) - Axis of symmetry: Vertical line $x = x_v$

## Vertex Form

Alternative representation: $f(x) = a(x - h)^2 + k$

- Vertex: $(h, k)$ - directly visible!
- Direction: $a$ (same as standard form)
- Advantage: Vertex immediately apparent
- Examples:
  - $f(x) = 2(x - 3)^2 + 5 \longrightarrow$ Vertex at $(3, 5)$, minimum
  - $g(x) = -(x + 4)^2 + 10 \longrightarrow$ Vertex at $(-4, 10)$, maximum

## Completing the Square

Convert from standard form to vertex form

Process:

1. Factor out $a$ from first two terms
2. Complete the square inside parentheses: add and subtract $\left(\frac{b}{2a}\right)^2$
3. Simplify to vertex form $a(x - h)^2 + k$

Example: Convert $f(x) = 2x^2 - 12x + 10$

1. Factor out 2: $f(x) = 2(x^2 - 6x) + 10$
2. Complete square: Need $\left(\frac{-6}{2}\right)^2 = 9$
3. Add/subtract 9: $f(x) = 2(x^2 - 6x + 9 - 9) + 10$
4. Factor perfect square: $f(x) = 2((x - 3)^2 - 9) + 10$
5. Distribute: $f(x) = 2(x - 3)^2 - 18 + 10$
6. Final form: $f(x) = 2(x - 3)^2 - 8 \longrightarrow$ Vertex at $(3, -8)$

# Business Optimization with Quadratic Functions

## Revenue Optimization (Price-Dependent Demand)

When demand depends on price: $Q = a - bp$

- Revenue: $R(p) = p \times Q = p(a - bp) = ap - bp^2$
- This creates a quadratic function in price
- Optimal price: $p^* = -\frac{a}{2(-b)} = \frac{a}{2b}$

## Profit vs Revenue Maximization

Important Distinction:

- Revenue maximization: Ignores costs, focuses on $R(p) = p \times Q(p)$
- Profit maximization: Includes costs, $\Pi(p) = R(p) - C(p)$
- Different optimal points: Revenue-maximizing price ≠ Profit-maximizing price

## Cost Functions (Quadratic)

Some costs increase quadratically with production:

- Formula: $C(x) = ax^2 + bx + c$
- Interpretation:
  - Fixed costs: $c$
  - Increasing marginal costs: $a > 0$
- Minimum cost production: $x^* = -\frac{b}{2a}$ (if $a > 0$)

## Profit Functions (Quadratic)

Common form: $P(x) = -ax^2 + bx + c$ (where $a > 0$)

- Opens downward: Has maximum profit
- Optimal production: $x^* = \frac{b}{2a}$
- Break-even points: Solve $P(x) = 0$

# Advanced Applications

## Area Optimization

Classic problem: Maximize area with constraint

Example: Rectangular storage with 200m fencing - One side against building (no fence needed) - Let $x$ = width, $y$ = length along building - Constraint: $2x + y = 200 \longrightarrow y =$

$200 - 2x$ - Area: $A(x) = x \cdot y = x(200 - 2x) = 200x - 2x^2$ - Maximum: $x^* = \frac{200}{2(2)} = 50m$ - Optimal dimensions: 50m × 100m = 5,000 m$^2$

### Projectile Motion in Business

Applications: Product launch campaigns, market penetration

- Formula: $h(t) = -\frac{g}{2}t^2 + v_0 t + h_0$
- Business analog: $A(t) = -at^2 + bt + c$ (awareness over time)
- Peak timing: $t^* = \frac{b}{2a}$

Example: Marketing Campaign - Awareness: $A(t) = -2t^2 + 24t$ - Peak at: $t = \frac{24}{2(2)} = 6$ weeks - Maximum awareness: $A(6) = 72$ points

### Market Analysis Steps

1. Identify the quadratic relationship (revenue, profit, cost)
2. Find the vertex using $x = -\frac{b}{2a}$
3. Calculate optimal value by substituting back
4. Check constraints (production limits, market capacity)
5. Consider break-even points (solve $f(x) = 0$)

### Key Business Insights

- Symmetric behavior: Equal results at equal distances from optimum
- Diminishing returns: Beyond optimal point, additional input reduces output
- Trade-offs: Revenue maximization vs profit maximization
- Constraints matter: Mathematical optimum may not be practically achievable

## Function Transformations

### Vertical Transformations

Moving graphs up or down

| Transformation | Formula | Effect | Business Example |
|---|---|---|---|
| Vertical Shift Up | $g(x) = f(x) + k$ (k > 0) | Entire graph moves up | Fixed cost increase |
| Vertical Shift Down | $g(x) = f(x) - k$ (k > 0) | Entire graph moves down | Government subsidy |
| Vertical Stretch | $g(x) = a \cdot f(x)$ (a > 1) | Graph becomes taller | Price increase across all products |

| Transformation | Formula | Effect | Business Example |
|---|---|---|---|
| Vertical Compression | $g(x) = a \cdot f(x)$ (0 < a < 1) | Graph becomes shorter | Discount pricing |
| Reflection over x-axis | $g(x) = -f(x)$ | Graph flips upside down | Revenue becomes loss |

Business Applications:

- Fixed Cost Changes: $C_{new}(x) = C(x) + k$ (rent increase of k)
- Percentage Markups: $R_{new}(x) = 1.2 \cdot R(x)$ (20% price increase)
- Tax Effects: $P_{net}(x) = 0.8 \cdot P(x)$ (20% tax rate)

## Horizontal Transformations

Moving graphs left or right

| Transformation | Formula | Effect | Business Example |
|---|---|---|---|
| Horizontal Shift Right | $g(x) = f(x - h)$ (h > 0) | Graph moves right | Market entry delay |
| Horizontal Shift Left | $g(x) = f(x + h)$ (h > 0) | Graph moves left | Early product launch |
| Horizontal Stretch | $g(x) = f(x/b)$ (b > 1) | Graph becomes wider | Extended product lifecycle |
| Horizontal Compression | $g(x) = f(bx)$ (b > 1) | Graph becomes narrower | Accelerated lifecycle |
| Reflection over y-axis | $g(x) = f(-x)$ | Graph flips left-right | Reverse time analysis |

Key Point: Horizontal transformations are counterintuitive!

- $f(x - 3)$ shifts RIGHT by 3 units
- $f(x + 3)$ shifts LEFT by 3 units
- $f(2x)$ compresses (makes narrower)
- $f(x/2)$ stretches (makes wider)

Business Applications:

- Seasonal Adjustments: $D(t) = f(t - 3)$ (peak shifts 3 months later)
- Market Speed Changes: $L(t) = f(2t)$ (2x faster product lifecycle)

- Time Scaling: Converting quarterly to monthly data

### Combining Transformations

Standard Order for: $g(x) = a \cdot f(b(x-h)) + k$

1. Horizontal shift by h
2. Horizontal stretch/compress by factor b
3. Vertical stretch/compress by factor a
4. Vertical shift by k

Example: Transform $f(x) = x^2$ to $g(x) = -2(x-3)^2 + 5$

1. Shift right 3: $(x-3)^2$
2. Stretch vertically by 2: $2(x-3)^2$
3. Reflect over x-axis: $-2(x-3)^2$
4. Shift up 5: $-2(x-3)^2 + 5$

## Reading Economic Graphs

### Key Features to Identify

Graph Analysis Checklist:

- Intercepts: Starting values, break-even points
- Slope/Rate of change: Marginal values, trends
- Maximum/Minimum: Optimal points, extremes
- Intersections: Equilibrium, equal values
- Shape: Linear, quadratic, exponential patterns
- Domain/Range: Feasible regions, constraints

### Business Graph Interpretation

Cost vs Revenue Analysis:

- y-intercept of cost: Fixed costs
- Intersection points: Break-even quantities
- Region between intersections: Profitable zone
- Maximum vertical distance: Optimal production level
- Slope comparisons: Marginal cost vs marginal revenue

Market Analysis:

- Supply/Demand intersection: Market equilibrium
- Shift patterns: Market changes over time
- Area under curves: Total consumer/producer surplus
- Steep vs flat curves: Price sensitivity (elasticity)

# Function Composition

## Understanding Composition

Composition models sequential processes: $(f \circ g)(x) = f(g(x))$

Key Points:

- Read as “f composed with g”
- Apply g first, then f to the result
- Output of g becomes input of f
- Order matters: $(f \circ g) \neq (g \circ f)$ usually
- Models multi-step business processes

Steps to Find Composition:

1. Calculate $g(x)$ first
2. Substitute result into $f$
3. Simplify the expression

## Common Business Compositions

- Manufacturing: Raw materials $\longrightarrow$ Components $\longrightarrow$ Finished products
- Finance: Local currency $\longrightarrow$ Foreign currency $\longrightarrow$ Investment returns
- Retail: Wholesale $\longrightarrow$ Retail price $\longrightarrow$ After-tax price
- Marketing: Leads $\longrightarrow$ Conversions $\longrightarrow$ Revenue

## Domain Considerations

When composing functions, the domain of $(f \circ g)$ is restricted by:

- Domain of g (inner function)
- Values of g(x) must be in domain of f (outer function)

# Inverse Functions

## What is an Inverse Function?

An inverse function reverses the original function

Definition: If $f(a) = b$, then $f^{-1}(b) = a$

- The inverse “undoes” what the original function does
- Notation: $f^{-1}(x)$ (read as “f inverse of x”)
- Not the same as $\frac{1}{f(x)}$ (reciprocal)

## Testing for Invertibility

A function has an inverse if and only if it’s one-to-one

Horizontal Line Test:

- Each horizontal line intersects the graph at most once

- Equivalently: each output comes from exactly one input
- For continuous functions: must be always increasing or always decreasing

### Finding Inverse Functions

Step-by-Step Process:

1. Replace $f(x)$ with $y$
2. Swap $x$ and $y$
3. Solve for $y$
4. Replace $y$ with $f^{-1}(x)$
5. Verify domain and range

### Business Applications of Inverse Functions

Common Business Inverses:

- Demand ↔ Price: $Q = 1000 - 20p \longrightarrow p = \frac{1000-Q}{20}$
- Cost ↔ Quantity: $C = 500 + 25x \longrightarrow x = \frac{C-500}{25}$
- Profit ↔ Sales: Find required sales for target profit
- Break-even Analysis: Reverse engineer production requirements

### Graphical Properties of Inverses

- Reflection property: Graph of $f^{-1}$ is reflection of $f$ over line $y = x$
- Domain and range swap: Domain of f = Range of $f^{-1}$
- Intersection points: $f$ and $f^{-1}$ intersect on line $y = x$

## Problem Solving

### Systematic Problem-Solving Approach

1. Identify the context: What business situation is modeled?
2. Determine function type: Linear, quadratic, composed, inverse?
3. Identify variables: Input (independent) vs output (dependent)
4. Check domain restrictions: Mathematical and business constraints
5. Write the function: Express relationship algebraically with units
6. Apply appropriate techniques: Vertex formula, composition, transformations
7. Verify results: Check if outputs make business sense
8. Consider practical constraints: Production limits, budget restrictions

### Common Pitfalls to Avoid

- Composition order: Remember $(f \circ g) \neq (g \circ f)$
- Domain issues: Always check for restrictions
- Units confusion: Track currency vs. quantity units carefully
- Inverse vs. reciprocal: $f^{-1}(x) \neq \frac{1}{f(x)}$
- Optimization assumptions: Verify that mathematical optimum is practical