Course Cheatsheet

Section 02: Equations & Problem-Solving Strategies

The IDEA Method

A systematic approach for solving word problems:

- Identify: What type of problem are we solving?
- Develop: Create a plan using appropriate methods
- Execute: Carry out the solution carefully
- Assess: Check your answer makes sense

Translating Words to Mathematics

English Phrase	Symbol	Example
"is", "equals"	=	"The cost is €50" → $C = 50$
"less than"	<	"x is less than 10" → $x < 10$
"at least"	≥	"at least 5 units" → $x \geq 5$
"at most"	≤	"at most 100" → $x \leq 100$
"increased by"	+	"price increased by €5" → $p + 5$
"decreased by"	-	"reduced by 20%" → $x - 0.2x$
"of", "times"	×	"30% of sales" → $0.3S$
"per"	÷	"cost per unit" → $\frac{\text{total cost}}{\text{units}}$

Business Vocabulary

Essential Terms:

- Revenue (R): Total income = Price × Quantity
- Cost (C): Fixed costs + Variable costs
- Profit (P): Revenue - Cost = R - C
- Break-even: When Revenue = Cost (Profit = 0)
- Margin: Profit as percentage of revenue

- Markup: Increase from cost to selling price

Linear Equations

Standard Form: $ax + b = c$

Solving Multi-Step Equations:

1. Clear fractions by multiplying by LCD
2. Expand parentheses using distributive property
3. Collect like terms (variables on one side, constants on other)
4. Isolate variable by dividing by coefficient
5. Verify by substituting back

Example with Fractions:

$$\frac{2x-1}{3} + \frac{x+2}{4} = 5$$

- LCD = 12
- Multiply through: $4(2x-1) + 3(x+2) = 60$
- Expand: $8x - 4 + 3x + 6 = 60$
- Solve: $11x = 58$, so $x = \frac{58}{11}$

Inequalities

> ! Key Rule
>
> When multiplying or dividing by a negative number, flip the inequality sign!

Example: $-2x > 6$

- Divide by –2: $x < -3$ (sign flipped!)

Solution Notation:

- $x < a$: interval $(-\infty, a)$
- $x \leq a$: interval $(-\infty, a]$
- $x > a$: interval (a, ∞)
- $a < x < b$: interval (a, b)

Systems of Linear Equations

2×2 Systems

Two Methods:

1. Substitution Method (best when one variable is isolated):

- Isolate one variable in one equation
- Substitute into the other equation
- Solve for remaining variable

- Back-substitute to find first variable

2. Elimination Method (best for symmetric systems):

- Align equations vertically
- Multiply to create opposite coefficients
- Add/subtract to eliminate one variable
- Solve for remaining variable

Three Possible Outcomes:

- Unique Solution: Lines intersect once (most common)
- No Solution: Parallel lines (inconsistent system)
- Infinite Solutions: Same line (dependent equations)

Quick Classification

For system $\{a_1x + b_1y = c_1 \quad a_2x + b_2y = c_2$:

- If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$: No solution
- If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$: Infinite solutions
- Otherwise: Unique solution

Quadratic Equations

Standard Form: $ax^2 + bx + c = 0$

The Discriminant

$\Delta = b^2 - 4ac$ tells us:

Δ Value	Solution Type	Graph Behavior
$\Delta > 0$ (perfect square)	Two rational solutions	Crosses x-axis twice
$\Delta > 0$ (not perfect square)	Two irrational solutions	Crosses x-axis twice
$\Delta = 0$	One repeated solution	Touches x-axis once
$\Delta < 0$	No real solutions	Doesn't touch x-axis

Three Solution Methods

1. Factoring (when Δ is a perfect square):

- Factor the quadratic
- Apply Zero Product Property: If $AB = 0$, then $A = 0$ or $B = 0$

2. Quadratic Formula (always works):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Completing the Square (useful for deriving vertex form):

- Move constant to right side
- Add $\left(\frac{b}{2a}\right)^2$ to both sides
- Factor left side as perfect square

Method Selection Guide

```
Calculate Δ = b² - 4ac
│
├─ Δ < 0 → No real solutions
├─ Δ = 0 → One solution: x = -b/(2a)
└─ Δ > 0 → Two real solutions
           └─ Is Δ a perfect square?
              ├─ YES → Try factoring first
              └─ NO → Use quadratic formula
```

Biquadratic Equations

Form: $ax^4 + bx^2 + c = 0$

Solution Strategy:

1. Let $u = x^2$
2. Solve $au^2 + bu + c = 0$ (quadratic in u)
3. Back-substitute: If $u = k$, then $x^2 = k$
4. Solve for x: $x = \pm\sqrt{k}$ (if $k \geq 0$)

Example: $x^4 - 5x^2 + 4 = 0$

- Let $u = x^2$: $u^2 - 5u + 4 = 0$
- Factor: $(u-1)(u-4) = 0$
- So $u = 1$ or $u = 4$
- Therefore: $x = \pm 1$ or $x = \pm 2$

Fractional (Rational) Equations

Key Steps:

1. Find domain restrictions (denominators ≠ 0)
2. Clear fractions by multiplying by LCD
3. Solve resulting equation
4. Check solutions against domain restrictions

Example: $\frac{2}{x-1} + \frac{3}{x+2} = 1$

- Domain: $x \neq 1, x \neq -2$
- LCD: $(x-1)(x+2)$
- Clear fractions: $2(x+2) + 3(x-1) = (x-1)(x+2)$
- Expand and solve: $x^2 - 4x - 5 = 0$
- Solutions must be checked against domain!

Cross Multiplication: For $\frac{a}{b} = \frac{c}{d}$, we get $ad = bc$

Radical Equations

Solution Strategy:

1. Isolate the radical term
2. Square both sides (or raise to appropriate power)
3. Solve resulting equation
4. CHECK ALL SOLUTIONS

> ! Really, don't forget checking the solutions
>
> Squaring can introduce extraneous solutions!

Example: $\sqrt{x+3} = x - 1$

- Square both sides: $x + 3 = (x-1)^2$
- Expand: $x + 3 = x^2 - 2x + 1$
- Rearrange: $x^2 - 3x - 2 = 0$
- Solutions must be checked in original equation!

Multiple Radicals:

- Isolate one radical at a time
- Square, simplify, repeat if necessary

Cubic Equations (Recap from Section 01)

Form: $ax^3 + bx^2 + cx + d = 0$

Solution Strategies:

1. Rational Root Theorem: Possible rational roots = $\pm\frac{\text{factors of } d}{\text{factors of } a}$

2. Special Forms:

- Sum of cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- Difference of cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

3. Factor by Grouping:

- Look for common factors in pairs of terms

Example: $x^3 - 7x^2 + 14x - 8 = 0$

- Test rational roots: Try $x = 1$: $1 - 7 + 14 - 8 = 0$ ✓
- Factor out $(x-1)$ using synthetic division
- Get: $(x-1)(x^2 - 6x + 8) = 0$
- Factor further: $(x-1)(x-2)(x-4) = 0$

Exponential Equations

Basic Strategies:

1. Same Base Method:

If $a^f(x) = a^{g(x)}$, then $f(x) = g(x)$

2. Logarithm Method:

For $a^x = b$, take log of both sides:

- $\log(a^x) = \log(b)$
- $x \log(a) = \log(b)$
- $x = \frac{\log(b)}{\log(a)}$

3. Substitution for Complex Forms:

For $4^x - 3 \cdot 2^x + 2 = 0$:

- Note: $4^x = (2^2)^x = (2^x)^2$
- Let $u = 2^x$: $u^2 - 3u + 2 = 0$

Mixed Base Systems:

For different bases, use logarithms strategically or look for relationships

Logarithmic Equations

Key Strategies:

1. Use Properties to Combine:

- Product: $\log_a(x) + \log_a(y) = \log_a(xy)$
- Quotient: $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$
- Power: $n \log_a(x) = \log_a(x^n)$

2. Convert to Exponential Form:

If $\log_a(x) = y$, then $a^y = x$

3. Domain Restrictions:

Always ensure arguments of logarithms are positive!

Example: $\log(x) + \log(x - 3) = 1$

- Domain: $x > 3$
- Combine: $\log(x(x - 3)) = 1$
- Convert: $x(x - 3) = 10$
- Solve: $x^2 - 3x - 10 = 0$
- Check domain restrictions!