Course Cheatsheet

Section 01: Mathematical Foundations & Algebra

Number Systems

The hierarchy of number systems:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{I} \subset \mathbb{R}$$

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ - Natural numbers
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ - Integers
- $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$ - Rational numbers
- $\mathbb{I}$ - Irrational numbers (cannot be expressed as fractions)
- $\mathbb{R}$ - Real numbers (all points on the number line)

Key Facts:

- Repeating decimals are rational: $0.\overline{36} = \frac{36}{99} = \frac{4}{11}$
- π, $\sqrt{2}$, $\sqrt{7}$ are irrational

Set Theory Essentials

Set Notation:

- Roster: $A = \{1, 2, 3, 4, 5\}$
- Set-builder: $B = \{x \in \mathbb{N} : x < 6\}$
- Interval: $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

Set Operations:

- Union: $A \cup B$ (elements in A or B)
- Intersection: $A \cap B$ (elements in A and B)
- Difference: $A \setminus B$ (elements in A but not in B)

Essential Symbols:

- $\in$ (element of), $\notin$ (not element of)
- $\subset$ (subset), $\subseteq$ (subset or equal)
- $\forall$ (for all), $\exists$ (there exists)
- $\Rightarrow$ (implies), $\Leftrightarrow$ (if and only if)

Commutative, Associative, and Distributive Laws

Commutative: $a + b = b + a$, $a \times b = b \times a$

Associative: $(a + b) + c = a + (b + c)$

Distributive: $a(b + c) = ab + ac$

Laws of Exponents

Rule	Formula	Example
Product	$a^m \cdot a^n = a^{m+n}$	$x^3 \cdot x^4 = x^7$
Quotient	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{x^5}{x^2} = x^3$
Power	$(a^m)^n = a^{mn}$	$(x^3)^2 = x^6$
Product Power	$(ab)^n = a^n b^n$	$(2x)^3 = 8x^3$
Quotient Power	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{x}{3}\right)^2 = \frac{x^2}{9}$

Special Values:

- $a^0 = 1$ (for $a \neq 0$)
- $a^1 = a$
- $a^{-n} = \frac{1}{a^n}$
- $a^{1/n} = \sqrt[n]{a}$

Scientific Notation

Format: $a \times 10^n$ where $1 \leq |a| < 10$

Examples:

- $56,700,000 = 5.67 \times 10^7$
- $0.00000423 = 4.23 \times 10^{-6}$

Operations:

- Multiply: $(3 \times 10^5) \times (2 \times 10^3) = 6 \times 10^8$
- Divide: $\frac{8.4 \times 10^7}{2.1 \times 10^4} = 4 \times 10^3$

Absolute Value

Definition: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Solving Equations: $|ax + b| = c$ has solutions:

- $ax + b = c$ and $ax + b = -c$

Inequalities:

- $|x| < a$ means $-a < x < a$
- $|x| > a$ means $x < -a$ or $x > a$

Basic Factorization

Common Factor: Always check first!

- $12x^3 - 18x^2 + 6x = 6x(2x^2 - 3x + 1)$

Difference of Squares:

- $a^2 - b^2 = (a + b)(a - b)$
- $x^2 - 9 = (x + 3)(x - 3)$

Perfect Square Trinomials:

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$

Advanced Factorization

AC Method: For $ax^2 + bx + c$ when $a \neq 1$

1. Find ac
2. Find factors of ac that sum to b
3. Rewrite middle term and group
4. Factor by grouping

Example: $6x^2 + 13x + 5$

- $ac = 30$, factors $(3, 10)$ sum to 13
- $6x^2 + 3x + 10x + 5 = (3x+5)(2x+1)$

Sum and Difference of Cubes:

- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Factoring by Grouping:

- Group terms with common factors
- Factor each group, then factor the common binomial

Roots and Radicals

Properties of Radicals:

Property	Formula	Example
Product	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt{12} = 2\sqrt{3}$
Quotient	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt{\frac{16}{4}} = 2$
Power	$\sqrt[n]{a^m} = a^{m/n}$	$\sqrt[3]{x^6} = x^2$

Sign Rules:

- Even roots: Always positive ($\sqrt{9} = 3$, not -3)
- Odd roots: Keep original sign ($\sqrt[3]{-8} = -2$)

Rationalizing Denominators:

- Simple: $\frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$
- Binomials: Use conjugates $(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c$

Substitution for Factorization

Strategy: Replace repeated expressions with a simpler variable to reveal hidden patterns.

Common Patterns:

- Quadratic in form: $x^4 - 13x^2 + 36$
  - Let $u = x^2$: $u^2 - 13u + 36 = (u-4)(u-9) = (x^2-4)(x^2-9)$
- Repeated expressions: $(2x+1)^2 - 3(2x+1) - 10$
  - Let $u = 2x+1$: $u^2 - 3u - 10 = (u-5)(u+2) = (2x-4)(2x+3)$
- Radical expressions: $x + 6\sqrt{x} + 8$
  - Let $u = \sqrt{x}$: $u^2 + 6u + 8 = (u+2)(u+4) = (\sqrt{x}+2)(\sqrt{x}+4)$

Steps:

1. Identify the repeated pattern
2. Substitute with a simple variable
3. Factor the resulting expression
4. Substitute back
5. Check if further factoring is possible

Logarithms

Definition: If $a^x = b$, then $\log_a(b) = x$

Basic Properties:

- $\log_a(1) = 0$ (since $a^0 = 1$)
- $\log_a(a) = 1$ (since $a^1 = a$)
- $\log_a(a^x) = x$ (inverse operations)
- $a^{\log_a(x)} = x$ (inverse operations)

Laws of Logarithms:

Rule	Formula	Example
Product	$\log_a(xy) = \log_a(x) + \log_a(y)$	$\log(20) = \log(4) + \log(5)$
Quotient	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$	$\log(\frac{100}{4}) = \log(100) - \log(4)$
Power	$\log_a(x^n) = n\log_a(x)$	$\log(8^3) = 3\log(8)$

Change of Base Formula:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} = \frac{\ln(x)}{\ln(a)} = \frac{\log(x)}{\log(a)}$$

Common Notations:

- $\log(x)$ means $\log_{10}(x)$ (common logarithm)
- $\ln(x)$ means $\log_e(x)$ (natural logarithm, $e \approx 2.718$)

Solving Exponential Equations:

- If bases are the same: $a^x = a^y \Rightarrow x = y$
- If bases differ: Take logarithms of both sides

Pascal's Triangle and Binomial Expansion

Pascal's Triangle:

```
Row 0:              1
Row 1:            1   1
Row 2:          1   2   1
Row 3:        1   3   3   1
Row 4:      1   4   6   4   1
Row 5:    1   5   10  10  5   1
```

Pattern: Each number = sum of the two numbers above it

Common Expansions:

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Binomial Theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ (later important!)

Compound Growth & Interest

Core Forms (from Sessions 01-03, 01-05, 01-06):

- (Discrete once per period) $A = P(1+r)^t$
  - P principal, r rate per period, t number of periods
- (Compounded n times per year) $A = P\left(1 + \frac{r}{n}\right)^{nt}$
  - r nominal annual rate, n compounding frequency (12 monthly, 4 quarterly, etc.)
- (Continuous compounding) $A = Pe^{rt}$