Mock Exam 05: Foundations through Differential Calculus

BFP Mathematics Course

Dr. Nikolai Heinrichs & Dr. Tobias Vlćek

Name: ______________________

Duration: 180 minutes

Total Points: 100

Permitted Aids:

- Calculator (non-programmable without graphing capabilities)
- Drawing instruments
- Monolingual dictionary
- No books, notes, or formula sheets

Instructions:

- Work through problems systematically, showing all steps
- Time yourself to practice exam conditions
- Check solutions afterward to identify areas needing review

Problem 1: Service Industry Cost Analysis [36 pts. total]

A consulting firm analyzes its cost structure for providing business advisory services. Financial analysis reveals the following information:

The fixed costs are 180 currency units (abbreviated as "CU" hereafter). At 2 quantity units of service (abbreviated as "Un" hereafter) the marginal costs are 12 CU. At 3 Un the curvature of the cost function changes sign. At 10 Un the total costs amount to 980 CU.

Market research shows that the demand for their services follows a linear function assuming a maximum price of 256 CU and a market saturation at 8 Un.

Part A: Function Development

a) Determine the cost function $K(x)$, assuming it is a polynomial of third degree. [13 pts.]

b) Determine the linear demand function $p(x)$. [3 pts.]

For verification purposes only:

$K(x) = 2x^3 - 18x^2 + 60x + 180$

$p(x) = -32x + 256$

c) Determine the revenue function $E(x)$ and show that it can be written as $E(x) = -32x^2 + 256x$. [2 pts.]

d) Show that the profit function is given by $G(x) = -2x^3 - 14x^2 + 196x - 180$. [2 pts.]

Part B: Optimization and Business Strategy

e) Prove that the break-even point is at 1 Un of produce. Explain the significance of this quantity for the company. [3 pts.]

f) Compute the maximum profit and prove it really is a maximum. [4 pts.]

g) Decide which price the company has to ask for in order to gain the maximum profit. [3 pts.]

h) Determine the marginal profit at the profit-maximizing production level. Interpret what this value tells us about the firm's pricing strategy. [2 pts.]

i) Compute the minimum variable cost per unit and the short-term lower limit price. [4 pts.]

Problem 2: Exponential Function Analysis [39 pts. total]

Let the function f be given by $f(x) = x \cdot e^{-x/2}, \quad x \in \mathbb{R}$

Part I: Basic Properties and Behavior

a) Determine the domain of the function f. [1 pt.]

b) Investigate the asymptotic behavior of f, and determine the x- and y-intercepts. [8 pts.]

c) Explain in complete sentences what the asymptotic behavior tells us about the long-term behavior of this function. [2 pts.]

Part II: Critical Analysis

d) Compute the first derivative $f'(x)$ using the product rule. [4 pts.]

e) Determine the nature (classify as local maximum, local minimum, or saddle point) and the coordinates of all stationary points (critical points where $f'(x) = 0$). [5 pts.]

f) Find the equation of the tangent line at the point $(2, \frac{2}{e})$. [4 pts.]

g) Compute the angle of intersection α between the tangent and the x-axis. [2 pts.]

h) Find the second derivative $f''(x)$ and determine where the function changes concavity (inflection points). [4 pts.]

For verification purposes only:

$f'(x) = e^{-x/2}(1 - x/2)$

Maximum at $x = 2$

i) Argue whether the following statements are true or false: [3 pts.]

(i) $f''(2) = 0$

(ii) In the interval $0 < x < 2$, the function f is concave down.

(iii) $f'(x)$ possesses a maximum at $x = 2$.

(iv) The tangential gradient of f at $x = 0$ is bigger than the slope of the secant in the interval $[0, 2]$.

j) Sketch the graph G_f in the interval $[-2; 10]$. Label stationary points, inflection points, and intercepts. [6 pts.]

Problem 3: Function Determination and Parameter Analysis [25 pts. total]

Part A: Revenue Function Determination

An e-commerce company's revenue function is modeled by a cubic polynomial $R(x) = ax^3 + bx^2 + cx + d$, where x represents thousands of customers served per month.

The following conditions are established from historical data:

- When serving zero customers, revenue is zero: $R(0) = 0$
- At 4 thousand customers, the revenue function has an inflection point with a value of 256 CU
- Revenue reaches a local maximum at 8 thousand customers

a) Translate each condition into a mathematical equation. Explain why the inflection point condition at $(4, 256)$ provides two equations. [4 pts.]

b) Set up and solve the complete system of equations to find a, b, c, and d. [8 pts.]

c) Verify that your function satisfies the conditions $R(4) = 256$ and $R''(4) = 0$. [2 pts.]

For verification purposes only:

$R(x) = -2x^3 + 24x^2$

Part B: Function Family Investigation

Consider the family of functions defined by:

$$f_a(x) = (x - a)^2 \cdot e^{-x}, \quad x \in \mathbb{R}, \quad a \in \mathbb{R}$$

d) Show that for all values of the parameter a, each function f_a has exactly one zero. State the coordinates of this zero in terms of a. [3 pts.]

e) Finding the local maximum:

(i) Show that the first derivative can be written as:

$$f_{a'}(x) = e^{-x}(x - a)(a + 2 - x)$$

[2 pts.]

(ii) Using the factored form above, find all critical points of f_a. Explain why $x = a$ is not a local extremum. [2 pts.]

(iii) Verify that the local maximum occurs at $x = a + 2$ and calculate $f_a(a + 2)$. [2 pts.]

f) The maximum value of f_a is $y_{max} = 4e^{-(a+2)}$.

(i) Calculate the maximum height when $a = 0$. [1 pt.]

(ii) For which value of a does the maximum height equal $4e^{-3}$? [1 pt.]

Appendix: Practice Materials (Not Part of Examination)

Grading Scale Reference

Grade	Points Required	Percentage	Self-Assessment
1 (Excellent)	91-100	91-100%	Outstanding
2 (Very Good)	77-90	77-90%	Strong understanding
3 (Good)	63-76	63-76%	Solid competence
4- (Pass)	45-62	45-62%	Meets requirements
5-6 (Fail)	0-44	0-44%	Needs review

Focus Areas by Score Range:

If you scored 0-44 points:

- Review fundamental concepts in Sections 01-04
- Practice basic derivative rules and function analysis
- Focus on problem setup and equation-solving techniques
- Consider additional practice with simpler problems first

If you scored 45-62 points:

- Strong foundation, but work on connecting concepts
- Practice optimization problems with economic interpretation
- Improve curve sketching and graphical analysis skills
- Review related rates systematic approach

If you scored 63-76 points:

- Good understanding overall
- Focus on excellence-level problems (parts f-h)
- Practice comprehensive function analysis
- Work on explaining business interpretations clearly

If you scored 77-90 points:

- Very strong preparation
- Fine-tune proof techniques and verification steps
- Practice time management for complex problems
- Review any specific topics where you lost points

If you scored 91-100 points:

- Excellent mastery
- Maintain your preparation level
- Help others understand difficult concepts
- Focus on exam strategy and time optimization

Topic Review Checklist

After completing this exam, identify which topics need review:

Problem 1 Topics:

- ☐ Function determination from conditions
- ☐ Setting up systems of equations from word problems
- ☐ Linear demand functions
- ☐ Revenue and profit function relationships
- ☐ Break-even analysis
- ☐ Optimization using derivatives
- ☐ Second derivative test for maxima/minima
- ☐ Marginal analysis and interpretation
- ☐ Minimum variable cost per unit (Betriebsminimum)
- ☐ Short-term lower limit price

Problem 2 Topics:

- ☐ Exponential functions and their properties
- ☐ Product rule for differentiation
- ☐ Limit analysis for asymptotic behavior
- ☐ Critical point identification and classification
- ☐ Tangent line equations at specific points
- ☐ Angle of intersection with coordinate axes
- ☐ Second derivative and concavity
- ☐ Inflection points
- ☐ True/False reasoning about derivative properties
- ☐ Curve sketching with all features
- ☐ Interpreting mathematical results in context

Problem 3 Topics:

- ☐ Cubic function determination from mixed conditions
- ☐ Recognizing when inflection points provide two equations
- ☐ Solving 4×4 systems of equations
- ☐ Function families with parameters
- ☐ Finding zeros of parametric functions
- ☐ Product rule with parametric expressions
- ☐ Extrema of parametric functions
- ☐ Evaluating parametric expressions at specific values

Remember: This practice exam is a learning tool. Use your performance to guide your study, not to judge your abilities!